

ON THE NON-VANISHING OF THE JACOBIAN IN
CERTAIN ONE-TO-ONE MAPPINGS

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THEOREM 1. *If $u(x, y)$ and $v(x, y)$ are harmonic, $u(0, 0) = v(0, 0) = 0$, and if there exists a neighborhood N_1 of the origin of the xy plane and a neighborhood N_2 of the origin of the uv plane such that $u(x, y)$ and $v(x, y)$ establish a mapping of N_1 onto N_2 which is one-to-one both ways, then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.*

PROOF. As the statement of Theorem 1 remains invariant under homogeneous linear transformations of the uv plane, we may assume, in the developments in polar coordinates for u and v , that

$$u = \sum_i^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta], \quad (a_i^2 + b_i^2 \neq 0),$$

$$v = \sum_k^{\infty} [A_n r^n \cos n\theta + B_n r^n \sin n\theta], \quad (A_k^2 + B_k^2 \neq 0),$$

that the positive index i does not exceed k , and that for $i = k$ we have $a_k B_k - A_k b_k \neq 0$. Considering the case $i = k$ first, we may, because of the invariance mentioned, assume

$$a_k = B_k = 1, \quad b_k = A_k = 0.$$

For small values of r , the auxiliary mapping,

$$\bar{u} = r^k \cos k\theta, \quad \bar{v} = r^k \sin k\theta,$$

can be continuously joined with the given one for $0 \leq t \leq 1$ by

$$u_t = r^k \cos k\theta + t \sum_{k+1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

$$v_t = r^k \sin k\theta + t \sum_{k+1}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The vector (u_t, v_t) thereby never differs by more than a vector of length $r^k/2$ from the vector (\bar{u}, \bar{v}) whose length is r^k . Thus

the index of the origin* in both fields (u, v) and (\bar{u}, \bar{v}) is the same, and as it is ± 1 for one-to-one mappings we conclude $i = k = 1$ and our theorem follows.

In the remaining case $i < k$ we consider the Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{r} \frac{\partial(u, v)}{\partial(r, \theta)}.$$

We find by elementary computation that the development of J starts with the terms of lowest degree in r

$$kir^{i+k-1}[(A_k a_i + B_k b_i) \sin(i - k)\theta + (a_i B_k - b_i A_k) \cos(i - k)\theta]$$

or, with a suitable angle θ_0 , with the term

$$kir^{i+k-1}(A_k^2 + B_k^2)^{1/2}(a_i^2 + b_i^2)^{1/2} \cos[(i - k)\theta - \theta_0].$$

Hence, for sufficiently small values of r , J assumes both positive and negative values, while in a one-to-one map the Jacobian, where it does not vanish, is of the same sign as the index of the map which is either $+1$ everywhere or -1 everywhere. Hence $i < k$ is impossible and our proof is completed.

THEOREM 2. *Let $u(x, y)$ and $v(x, y)$ be analytic functions of x and y in a neighborhood N_1 of the origin of the xy plane which they map onto a neighborhood N_2 of the origin of the uv plane in a one-to-one correspondence. Suppose, moreover, that u and v are solutions of the following equations:*

$$(1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + b \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \\ + c \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + d \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) = 0,$$

$$(2) \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + A \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + B \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \\ + C \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + D \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) = 0,$$

* For the topological notions used, see W. Fenchel, *Elementare Beweise und Anwendungen einiger Fixpunktsätze*, Matematisk Tidsskrift, (B), 1932, p. 66.

in which $a(u, v)$, $b(u, v)$, \dots , $D(u, v)$ are analytic functions of u and v defined for (u, v) in N_2 . Then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.

PROOF. Suppose, without loss of generality, that the power series for u and v in x and y start with non-vanishing terms of i th and k th degree, respectively, and that furthermore $i \leq k$, $i > 0$. Then the terms of i th degree in $u(x, y)$ form a harmonic function, because this is trivial for $i = 1$, and for $i > 1$ the only terms in the development of the left hand of (1) of degree $i - 2$ are furnished by those of $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2$. Introduce $V(x, y) = v(x, y) - F(u)$, where F is analytic in u and $F(0) = 0$. Instead of (1) and (2) we find

$$(3) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [a(u, V + F(u)) + F'(u)b(u, V + F(u)) \\ + F'^2(u)c(u, V + F(u))] \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + \dots = 0,$$

$$(4) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + F'(u) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ + [A(u, V + F(u)) + F'(u)B(u, V + F(u)) + F'^2(u)C(u, V + F(u)) \\ + F''(u)] \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + \dots = 0,$$

and, by substitution of (3) in (4),

$$(5) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + [A(u, V + F(u)) + F'B + F'^2C + F'' - F'a \\ - F'^2b - F'^3c] \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] + \dots = 0,$$

where the omitted terms are linear in

$$\frac{\partial u}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial V}{\partial y}, \quad \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2, \quad \text{and} \quad \frac{\partial(u, V)}{\partial(x, y)}.$$

Now notice that V vanishes for $x = y = 0$ and that in its development the lowest degree K of non-vanishing terms still is $\geq i$.

Let us distinguish two cases, $k > i$ and $k = i$. For $k > i$, we con-

struct a function $F(u)$, solution of the ordinary differential equation of second order

$$(6) \quad A(u, F(u)) + F'(u)B(u, F(u)) \\ + F'^2C - F'a - F'^2b - F'^3c + F'' = 0,$$

vanishing with its first derivative for $u=0$. We have $k > i$, as $F(u)$, considered as function of x and y starts with terms of degree $\geq 2i$. Equation (5) shows now a coefficient of $(\partial u/\partial x)^2 + (\partial u/\partial y)^2$ whose development in u and V has *all terms divisible* by V . Thus the x, y series for the left hand of (5) obtains all terms of degree $K-2$ from the expression $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2$, whence we conclude that in V the terms of lowest order K form a harmonic function. As in the proof of Theorem 1, we conclude that the Jacobian $\partial(u, V)/\partial(x, y)$, which equals $\partial(u, v)/\partial(x, y)$, would assume both positive and negative values in every neighborhood of the origin, which leads to a contradiction. Thus $i < k$ is impossible.

In the case $i = k$ we again choose $F(u)$ as solution of (6), vanishing for $u=0$, but we try to determine $F'(u)$ in such a way that k results $> i$. If this were possible, we could apply the same argument as at the end of the last paragraph, and find the same contradiction. Hence, should $i = k = 1$, we would have the desired inequality

$$\frac{\partial(u, V)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)} \neq 0$$

at the origin. If, however, $i = k > 1$, we may determine the linear term of $F(u)$ such that for polar coordinates in the xy plane and suitable determination of $\theta = 0$ we have

$$u = \text{const. } r^k \cos k\theta + r^{k+1}(\dots), \\ V = \text{const. } r^k \sin k\theta + r^{k+1}(\dots).$$

This leads, by the same reasoning as in the proof of Theorem 1, to the conclusion that the index of the vector field (u, V) at the origin is $k \neq 1$. But together with the functions $u(x, y), v(x, y)$, the functions $u(x, y), V(x, y) = v(x, y) - F(u)$ also establish a one-to-one mapping of the xy plane, because the correspondence between the uv plane and the uV plane evidently is one-to-one. Hence the index ought to be ± 1 , which completes our proof.

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