The paradox stated above is a particular case of Theorem 10, and therefore requires no further proof.

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THE BETTI NUMBERS OF CYCLIC PRODUCTS

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1. Introduction. In a recent paper† M. Richardson has discussed the symmetric product of a simplicial complex and has obtained explicit formulas for the Betti numbers of the two- and three-fold products. Acting on a suggestion of Lefschetz, we define a more general type of topological product and apply Richardson’s methods to compute the Betti numbers of a certain one of these, the “cyclic” product.

2. Basis for m-Cycles of General Products. Let S be a topological space and G a group of permutations on the numbers 1, · · · , n. The product of S with respect to G, G(S), is the set of all n-tuples \((P_1, \cdots, P_n)\) of points of S, where \((P_{i_1}, \cdots, P_{i_n})\) is to be regarded as identical with \((P_1, \cdots, P_n)\) if and only if the permutation \((i_1; \cdots; i_n)\) is an element of G. A neighborhood of \((P_1, \cdots, P_n)\) is the set of all points \((Q_1, \cdots, Q_n)\) for which \(Q_i\) belongs to a fixed neighborhood of \(P_i\). It is not difficult to verify that the

† M. Richardson, On the homology characters of symmetric products, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. We shall refer to this paper as R.
Hausdorff axioms hold for this definition of neighborhood, and hence that \( G(S) \) is a topological space. In particular, if \( G \) is the identity or the symmetric group, \( G(S) \) is, respectively, the direct or the symmetric product of \( S \). If \( G \) is the cyclic group on \( n \) elements we shall call \( G(S) \) the \( n \)-fold cyclic product of \( S \).

The space \( G(S) \) can be obtained in another manner. Let \( S^n \) denote the \( n \)-fold direct product of \( S \). Then each element \((i_1, \ldots, i_n)\) of \( G \) gives rise to an automorphism of \( S^n \) which carries \((P_1, \ldots, P_n)\) into \((P_{i_1}, \ldots, P_{i_n})\). By identifying points which are images of each other under the group of automorphisms we evidently obtain a space homeomorphic to \( G(S) \).

Now let \( K \) be a simplicial complex, \( K^n \) its direct product, and \( k = G(K) \) its product with respect to the group \( G \) of degree \( n \) and order \( r \). We then have \( r \) automorphisms \( T_\lambda \) of \( K^n \), and a continuous, single-valued transformation \( \Lambda \) of \( K^n \) into \( k \), such that\( {\dagger} \)

\[
\Lambda T_\lambda = \Lambda.
\]

Richardson has shown\( \ddagger \) that \( K^n \) and \( k \) can be subdivided into simplexes in such a fashion that the transformations \( T_\lambda \) and \( \Lambda \) are simplicial. We can therefore operate with them on chains of \( K^n \). If \( E \) and \( e \) are simplexes of \( K^n \) and \( k \), respectively, such that \( e = \Lambda E \), we define the operator \( \Lambda' \) by \( \Lambda' e = \sum_\lambda T_\lambda E \). We have then

\[
\Lambda\Lambda' e = re,
\]

\[
\Lambda'\Lambda E = \sum_\lambda T_\lambda E.
\]

We also find that \( T_\lambda \), \( \Lambda \), and \( \Lambda' \) preserve boundaries and hence homologies.

The principal theorem of Richardson,\( \S \) concerning the Betti numbers of \( k \), is stated in terms of matrices. For actual computation we find it easier to work with the cycles themselves, and so we shall state and prove the theorem in a slightly different form.

\( {\dagger} \) In the expression for the product of two transformations, the transformation represented by the right-hand symbol is to be applied first.

\( \ddagger \) R, pp. 51 and 53.

\( \S \) R, p. 52.
THEOREM 1. Let \( \{ \Gamma^i \} \) be an independent basis, with respect to homology, for \( m \)-cycles, with rational coefficients, of \( K^n \), such that \( T_\lambda \Gamma^i = \pm \Gamma^i \), \( (\lambda = 1, \cdots, r) \); and let \( \{ \Gamma^a \} \) be a maximal subset of \( \{ \Gamma^i \} \) such that

\[
\begin{align*}
(a) & \quad T_\lambda \Gamma^a \neq \pm \Gamma^a, \\
(b) & \quad T_\lambda \Gamma^a \neq - \Gamma^a,
\end{align*}
\]

for any \( \lambda \). Then \( \{ \gamma^a \} = \{ \Lambda \Gamma^a \} \) is an independent basis with respect to homology for the \( m \)-cycles of \( k \).

PROOF. (i) The \( \gamma^a \) are independent. For suppose that we have

\[
\sum a x_a \gamma^a \sim 0,
\]

that is, \( \sum a x_a \Lambda \Gamma^a \sim 0 \). Then

\[
\lambda' \sum_a x_a \Lambda \Gamma^a = \sum_a x_a \lambda' \Lambda \Gamma^a = \sum_{a, \lambda} x_a T_\lambda \Gamma^a \sim 0,
\]

by (3). Now if \( T_\lambda \Gamma^a = \epsilon \Gamma^i \), \( \epsilon = \pm 1 \), we cannot have \( T_\mu \Gamma^a = - \epsilon \Gamma^i \), for this would imply

\[
T_\mu^{-1} T_\lambda \Gamma^a = \epsilon T_\mu^{-1} \Gamma^i = - \epsilon \Gamma^a = - \Gamma^a,
\]

contrary to condition (b). Similarly, from (a), we cannot have \( T_\mu \Gamma^a = \pm \Gamma^a \), \( \beta \neq \alpha \). Hence with each such \( \Gamma^i \) there is associated an \( \epsilon_i \), a \( \Gamma^a \), and \( s_i \) values of \( \lambda \) for which \( T_\lambda \Gamma^a = \epsilon_i \Gamma^i \). If the last homology is now written in terms of the basis \( \{ \Gamma^i \} \), the coefficient of \( \Gamma^i \) will be \( \epsilon_i s_i x_a \). Since the \( \Gamma^i \) are independent, \( \epsilon_i s_i x_a = 0 \), and therefore every \( x_a = 0 \).

Use was made of the properties of the rational coefficients only in the last step of each part of the proof. Now the \( s_i \) introduced in (i) are factors of \( r \), for the \( T_\lambda \) for which \( T_\lambda \Gamma^a = \epsilon_i \Gamma^i \) evidently form a coset of the subgroup which leaves \( \Gamma^a \) invariant. It follows that the theorem will hold for any coefficient group in which each element has a unique \( r \)th part; in particular for the group of residues modulo a number prime to \( r \).

(ii) \( \{ \gamma^a \} \) is a basis. We note first that since the set \( \{ \Gamma^a \} \) is maximal every \( \Gamma^i \) is of one of the two forms \( T_\lambda \Gamma^a \) or \( \Gamma^i \), where for each \( j \) there is a \( \lambda_j \) such that \( T_{\lambda_j} \Gamma^i = - \Gamma^i \). Also, \( \Lambda \Gamma^i = \Lambda T_{\lambda_j} \Gamma^i = - \Lambda \Gamma^i \), so that \( \Lambda \Gamma^i = 0 \). Now if \( \gamma \) is any \( m \)-cycle of \( k \), \( \Lambda' \gamma \) is an \( m \)-cycle of \( K^n \), and so

\[
\Lambda' \gamma \sim \sum_i x_i \Gamma^i = \sum_{a, \lambda} x_{a \lambda} T_\lambda \Gamma^a + \sum_j x_j \Gamma^j.
\]
Hence
\[ \Lambda\Lambda' \gamma = r\gamma \sim \sum_{a, \lambda} x_{a\lambda} \Lambda T_{a} \Gamma^a + \sum_{j} x_{j} \Lambda \tilde{T}^j = \sum_{a, \lambda} x_{a\lambda} \gamma^a, \]
by (2) and (1). That is,
\[ \gamma \sim \sum_{a, \lambda} \frac{x_{a\lambda}}{r} \gamma^a, \]

3. **Betti Numbers of Cyclic Products.** Keeping the notation as before, we let \( G \) be the cyclic group on \( n \) elements. To compute the \( m \)th Betti number of the cyclic product \( k \) we must count the number of \( m \)-cycles \( \Gamma^a \). A basis of the type \( \{ \Gamma^i \} \) used in the theorem is obtained by taking all cycles of the form
\[ C_{m_1} \times \cdots \times C_{m_n}, \quad m_1 + \cdots + m_n = m, \]
\( C_{m_i} \) being a member of a basis of \( m_i \)-cycles of \( K \).

Following Richardson’s procedure, we obtain
\[ T_{\lambda}(C_{m_1} \times \cdots \times C_{m_{\lambda}} \times C_{m_{\lambda+1}} \times \cdots \times C_{m_n}) \]
\[ = (-1)^{m_{\lambda+1}} \Gamma_{\lambda} \times \cdots \times C_{m_n} \times C_{m_1} \times \cdots \times C_{m_\lambda}, \]
where
\[ \epsilon_1 = m_1m_2 + \cdots + m_{\lambda}m_n = m_1(m - m_1) = mm_1 - m_1^2 \]
\[ = mm_1 - m_1 \quad (\text{mod } 2) \]
\[ = (m - 1)m_1, \]
and by induction
\[ \epsilon_{\lambda} \equiv (m - 1)(m_1 + \cdots + m_{\lambda}) \quad (\text{mod } 2). \]

Let \( q \) be a factor of \( n \), \( n = qs \), and consider all \( \Gamma^i \) which are invariant, to within change of sign, under \( G_q \), the cyclic subgroup of \( G \) of order \( q \). They necessarily have the form
\[ \Gamma_q = (C_{m_1} \times \cdots \times C_{m_s}) \times (C_{m_1} \times \cdots \times C_{m_s}) \times \cdots \times (C_{m_1} \times \cdots \times C_{m_s}), \]
there being \( q \) identical sets of factors. We must have \( q(m_1 + \)

\[ ^{\dagger} S. Lefschetz, \text{Topology, p. 228.} \]
\[ \cdots + m_a = m; \] that is, to have a \( \Gamma_q \), \( q \) must be a factor of \( m \) and hence of \( (m, n) \), the highest common factor of \( m \) and \( n \). If \( t \) is a proper multiple of \( q \) and a factor of \( (m, n) \), it is easily seen that a \( \Gamma_t \) is also a \( \Gamma_q \). We denote by \( \Gamma_q^* \) any \( \Gamma_q \) which is not such a \( \Gamma_t \), and by \( A_{m, q} \) the number of \( \Gamma_q^* \). The total number of \( \Gamma_q \) is then \( \sum_t A_{m, t} \), the summation being over all values of \( t \) which are multiples of \( q \) and factors of \( (m, n) \). But the number of \( \Gamma_q \) is evidently equal to the number of possible combinations of the form \( C_{m_1} \times \cdots \times C_{m_s}, m_1 + \cdots + m_s = m/q \), and this is exactly \( R_{m/q}(K^*) \). Hence

\[ \sum_t A_{m, t} = R_{m/q}(K^*), \]

and from these equations we can obtain the \( A_{m, q} \) step by step starting with \( q = (m, n) \), or directly by the use of the Dedekind inversion formula.

Now

\[ T_s \Gamma_q = (-1)^{(m-1)(m_1+\cdots+m_a)} \Gamma_q = (-1)^{(m-1)m/q} \Gamma_q, \]

and so if \( m \) is even and \( m/q \) is odd, \( \Gamma_q \) is a cycle of the type \( \Gamma^i \) of Theorem 1 and is not counted among the \( \Gamma^a \). We therefore put

\[ B_{m, q} = \begin{cases} 0, & \text{if } m \text{ is even and } m/q \text{ is odd,} \\ A_{m, q}, & \text{otherwise.} \end{cases} \]

Consider the \( s \) cycles \( \Gamma_q^*, T_1 \Gamma_q^*, \cdots, T_{s-1} \Gamma_q^* \). If any two of these are equal, say \( T_i \Gamma_q^* = T_j \Gamma_q^*, (i > j) \), then \( \Gamma_q^* \) is invariant, to within change of sign, under the subgroup generated by \( T_j^{-1}T_i = T_{i-j} \), and hence under the minimal subgroup containing \( G_q \) and \( T_{i-j} \). Since \( i-j < s \), \( T_{i-j} \) is not an element of \( G_q \) and therefore this subgroup is a \( G_t \) with \( t \) a proper multiple of \( q \), contrary to the definition of \( \Gamma_q^* \). It follows that there are exactly \( s = n/q \) distinct transforms of each of the \( B_{m, q} \) cycles \( \Gamma_q^* \), and so we can pick out \( (q/n)B_{m, q} \) of the \( \Gamma_q^* \) which are not transformable into one another and which can therefore be included among the \( \Gamma^a \) of Theorem 1. Since the cycles \( \Gamma_q^* \) for different values of \( q \) are not transformable into one another and since every \( \Gamma^i \) is a \( \Gamma_q^* \) for some \( q \), we have the following result.
Theorem 2.

\[ R_m(k) = \frac{1}{n} \sum_q qB_{m,q}, \]

the summation being over all factors of \((m, n)\).

The following special cases may be of interest.

Corollary 1. If \(n\) is an odd prime

\[ R_m(k) = \begin{cases} 
(1/n)R_m(K^n), & \text{if } (m, n) = 1, \\
(1/n)[R_m(K^n) - R_s(K)] + R_s(K), & \text{if } m = ns. 
\end{cases} \]

Corollary 2. If \(p\) is an odd prime and \(n = p^\alpha, m = p^\beta m_1, (m_1, p) = 1, \) and \(\gamma = \min \alpha, \beta, \)

\[ R_m(k) = \frac{p - 1}{n} \left[ \frac{1}{p - 1} R_m(K^n) + \sum_{i=1}^{\gamma} p^{i-1}R_{m/p^i}(K^{n/p^i}) \right]. \]

Corollary 3. If \(R_0(K) = 1,\) then \(R_1(k) = R_1(K).\)

4. Remark. The methods used on the cyclic product can evidently be used to compute the Betti numbers of a product with respect to an arbitrary group. In general, however, the resulting formulas are too complicated to be of interest.

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