ON THE DETERMINATION OF EARTH CONDUCTIVITY FROM OBSERVED SURFACE POTENTIALS*

BY R. E. LANGER

If a direct current is supplied to the earth through a small electrode, a distribution of electrical potentials over the surrounding surface of the earth is produced. The variation of this potential with the distance from the electrode depends upon the manner in which the conductivity of the earth below the surface varies with the depth, the calculation of the potential when this conductivity is known being possible by familiar methods. From the standpoint of geophysics, however, it is most frequently the converse problem which is important. The surface potential is accessible, and is therefore at least theoretically measurable. From it the unknown subterranean conductivity is to be deduced. The discussions of this problem which are to be found in the geophysical literature have almost without exception been based upon considerations of the trial and error type. The potentials associated with a suitable set of hypothesized conductivities are computed, and by *fitting* observed potentials to these, as best may be, inferences about the unknown conductivity are sought.

A direct method, on the other hand, has been given recently by L. B. Slichter† and the present author.‡ It is assumed (as has generally been done) that the conductivity is a function only of the depth, and the Taylor's series expansion of this function is deduced from the surface potentials. The problem may, therefore, be regarded as solved whenever the conductivity function is one which is represented, at least to the depth in question, either by its Taylor's series directly or by analytic extensions of the same. The present paper is intended to extend the deductions to the case in which either the conductivity

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† L. B. Slichter, *The interpretation of the resistivity prospecting method for horizontal structures*, Physics, vol. 4 (1933), p. 307.

‡ R. E. Langer, *An inverse problem in differential equations*, this Bulletin, vol. 39 (1933), p. 814.

or its derivative shows a discontinuity within the depth range to be explored. Such discontinuities are evidently to be associated with stratifications of the earth's crust, and for their analysis the previous method, based as it is upon the use of power series, obviously requires extension. It is to be shown how such extension can be made, namely, how both the location and magnitude of the first discontinuity may be determined.

If the region of the earth in proximity with the electrode is idealized as an infinite half-space, the line of the electrode is a line of symmetry, and the cylindrical coordinates (ρ, x) (radial distance from the electrode, and depth) are clearly advantageous. Referred to them, the potential ϕ satisfies the differential equation

$$\sigma(x)\left\{\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\,\frac{\partial\phi}{\partial\rho}+\frac{\partial^2\phi}{\partial x^2}\right\}+\frac{d\sigma(x)}{dx}\cdot\frac{\partial\phi}{\partial x}=0, \tag{1}$$

in which $\sigma(x)$ represents the unknown conductivity.* The analysis of this equation may be made along familiar lines. The substitution $\phi = U(\rho)\cdot u(x)$ leads to a Bessel's equation for the radial component $U(\rho)$, and gives for the depth component $u(x)$ the differential equation

$$u'' + \frac{\sigma'(x)}{\sigma(x)}\,u' - \lambda^2 u = 0, \tag{2}$$

in which λ is a positive parameter. The conditions that the potential be everywhere finite and non-increasing, and that both the potential and the current be continuous, dictate for $U(\rho)$ the choice $J_0(\lambda\rho)$. For $u(x)$ they determine that continuous solution $u_0(x, \lambda)$ of the equation (2), which for every positive λ is a non-increasing function of x, and for which $\sigma(x)u_0'(x, \lambda)$ is continuous. The existence of just one such solution may be inferred from the fact that $\sigma(x)$, by its nature, is positive for all values of x. Finally the condition that $\partial\phi/\partial x$ be zero at the earth's surface, except at the electrode, may be made to yield the formula

$$\phi(\rho, x) = \frac{-c}{2\pi\sigma(0)}\int_0^\infty \frac{\sin\lambda a}{a}\,J_0(\lambda\rho)\,\frac{u_0(x, \lambda)}{u_0'(0, \lambda)}\,d\lambda,$$

* Many details which are here omitted are to be found in the papers cited.

in which c represents the current and a the radius of the electrode. If the function $\Omega(\lambda)$ is now defined in terms of the solution $u_0(x, \lambda)$ specified above, by means of the formula

$$\Omega(\lambda) = \frac{-\lambda u_0(0, \lambda)}{u_0'(0, \lambda)}, \tag{3}$$

the surface potential is clearly given by

$$\phi(\rho, 0) = \frac{c}{2\pi\sigma(0)} \int_0^\infty \Omega(\lambda) \frac{\sin \lambda a}{\lambda a} J_0(\lambda\rho) d\lambda.$$

Since this formula is invertible into

$$\Omega(\lambda) = \frac{2\pi a \lambda^2 \sigma(0)}{c \sin \lambda a} \int_0^\infty \phi(\rho, 0) J_0(\lambda\rho)\rho d\rho,$$

it is clear that a knowledge of the surface potential is co-extensive with a knowledge of $\Omega(\lambda)$. The latter may, therefore, be used in place of the former as a basis for the further considerations.

From the asymptotic expansion of the function $\Omega(\lambda)$ a power series in x which is formally the Taylor's series of $\sigma'(x)/\sigma(x)$ is deducible. This series, together with such analytic extensions of it as may exist, defines a function $s(x)$ over an interval, say $0 \leqq x \leqq H$, and from this in turn a function $\sigma_1(x)$ is determined through the formulas

$$\sigma_1(x) = \begin{cases} \sigma(0) e^{\int_0^x s(x) dx}, & \text{for } 0 \leqq x \leqq H, \\ \sigma_1(H), & \text{for } H < x. \end{cases} \tag{4}$$

The possibility or the feasibility of the computation of $\sigma_1(x)$ may be regarded as the factor limiting the constant H, and H represents the depth to which the "exploration" applies. It will be clear at once that if the true conductivity $\sigma(x)$ is representable over the interval $(0, H)$ by means of power series, it necessarily coincides with $\sigma_1(x)$, and its determination over the interval is in consequence completely covered in the discussions cited. In proceeding, therefore, it will be assumed that the coincidence of $\sigma(x)$ and $\sigma_1(x)$ ceases before the depth H is reached, that is, specifically,

$$\begin{aligned} \sigma(x) &= \sigma_1(x), \qquad \text{for } 0 \leqq x < h < H, \\ \sigma(h+) &= \mu\sigma(h-), \qquad \sigma'(h+) = \nu\sigma'(h-), \end{aligned} \tag{5}$$

the constants μ, ν, being positive and not both unity. The solution $u_0(x, \lambda)$ by virtue of its specification then satisfies in particular the relations

$$(6) \qquad u_0(h-, \lambda) = u_0(h+, \lambda), \qquad u_0'(h-, \lambda) = \mu u_0'(h+, \lambda),$$

and the entire problem crystallizes into the determination of the constants h, μ, and ν.

Let the function $\sigma_1(x)$ be looked upon now as a hypothetical conductivity. Then, in the manner outlined above it is associated with the function

$$(7) \qquad \Omega_1(\lambda) = \frac{-\lambda v_0(0, \lambda)}{v_0'(0, \lambda)},$$

with $v_0(x, \lambda)$ representing the non-increasing solution of the differential equation

$$(8) \qquad v'' + \frac{\sigma_1'(x)}{\sigma_1(x)} v' - \lambda^2 v = 0.$$

Since $\sigma_1(x)$ is continuous, the same is to be the case with $v_0(x, \lambda)$ and $v_0'(x, \lambda)$, and since $\sigma_1(x)$ is known the function (7) is computable.

Let $v_1(x, \lambda)$, $v_2(x, \lambda)$ designate the principal solutions of the differential equation (8) relative to the origin, that is, those for which

$$v_1(0, \lambda) = 0, \qquad v_1'(0, \lambda) = 1,$$
$$v_2(0, \lambda) = 1, \qquad v_2'(0, \lambda) = 0.$$

Then the functions (3) and (7) are expressible in the forms

$$(9) \qquad \Omega(\lambda) = \frac{\lambda W(u_0, v_1)}{W(u_0, v_2)}, \qquad \Omega_1(\lambda) = \frac{\lambda W(v_0, v_1)}{W(v_0, v_2)},$$

in which W denotes the Wronskian, that is,

$$W(f, \psi) = f(x)\psi'(x) - f'(x)\psi(x),$$

and in which any x on the range $(0, h)$ will serve in the evaluation of $\Omega(\lambda)$, while any x on $(0, H)$ serves in the case of $\Omega_1(\lambda)$. To verify these assertions it is necessary only to observe: firstly, that $v_1(x, \lambda)$, $v_2(x, \lambda)$ are also solutions of the differential equation (2) when x is on $(0, h)$; secondly, that the formulas (9) are obviously correct when the right-hand members are evaluated at the origin; and lastly, that the ratios of Wronskians involved are constants as to x on the intervals stated. Since

$$W(v_2, v_1) = \frac{\sigma(0)}{\sigma_1(x)} \quad \text{on } (0, H),$$

it is found that

$$\Omega_1(\lambda) - \Omega(\lambda) = \left.\frac{\lambda\sigma(0)W(u_0, v_0)}{\sigma(h-)W(u_0, v_2)W(v_0, v_2)}\right]_{x=h-}. \tag{10}$$

The evaluation of the right-hand member of this relation hinges now upon a determination of the functional forms of the several solutions which are involved.

When λ is large and x is on an interval on which $\sigma(x)$ is continuous the solutions of the differential equation (2) are known to admit of asymptotic representations with respect to λ. Specifically they may be represented in the form

$$u_j(x, \lambda) = \alpha_j F_+(x, \sigma) + \beta_j F_-(x, \sigma),$$

in which α_j, β_j are appropriate constants, while

$$F_\pm(x, \sigma) = \left[\frac{\sigma(0)}{\sigma(x)}\right]^{1/2} e^{\pm\lambda x}\left\{1 \pm \frac{\tau(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right\},$$

with

$$\tau(x) = \int_0^x \left\{\frac{\sigma''}{4\sigma'} - \frac{\sigma'^2}{8\sigma^2}\right\} dx.$$

Moreover,

$$F_\pm(0, \sigma) = 1,$$

$$F_\pm'(x, \sigma) = \left\{\pm\lambda - \frac{\sigma'(x)}{2\sigma(x)} + O\left(\frac{1}{\lambda}\right)\right\} F_\pm(x, \sigma).$$

Under corresponding conditions the solutions of the differential equation (8) may, of course, be similarly written

$$v_j(x, \lambda) = \gamma_j F_+(x, \sigma_1) + \delta_j F_-(x, \sigma_1).$$

Consider now any interval, $h \leqq x \leqq h+\epsilon$, on which $\sigma(x)$ is continuous. On this, as on any interval, $u_0(x, \lambda)$ is non-increasing. Since $F_+(x, \sigma)$ is an increasing function, and since it dominates the function $F_-(x, \sigma)$ for any $x>0$, it must be inferred that the coefficient α_0 is sufficiently small to make the term in question asymptotically negligible. Thus

$$u_0(x, \lambda) = F_-(x, \lambda) \quad \text{on} \quad (h, h+\epsilon),$$

and hence

$$u_0'(h+, \lambda) = \left\{-\lambda - \frac{\sigma'(h+)}{2\sigma(h+)} + O\left(\frac{1}{\lambda}\right)\right\} u_0(h+, \lambda).$$

In virtue of the relations (5) and (6) this may be written

$$\text{(11a)} \quad u_0'(h-, \lambda) = \left\{-\mu\lambda - \frac{\nu\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} u_0(h-, \lambda).$$

An entirely similar consideration, in which it is recalled, however, that $\sigma_1(x)$, $v_0(x, \lambda)$, and $v_0'(x, \lambda)$ are continuous, leads to the corresponding formula

$$\text{(11b)} \quad v_0'(h-, \lambda) = \left\{-\lambda - \frac{\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} v_0(h-, \lambda).$$

Finally the relation

$$\begin{aligned} v_2(x, \lambda) = \frac{1}{2}&\left\{1 + \frac{\sigma'(0)}{2\lambda\sigma(0)} + O\left(\frac{1}{\lambda^2}\right)\right\} F_+(x, \sigma_1) \\ &+ \frac{1}{2}\left\{1 - \frac{\sigma'(0)}{2\lambda\sigma(0)} + O\left(\frac{1}{\lambda^2}\right)\right\} F_-(x, \sigma_1), \end{aligned}$$

is directly verifiable on the basis of the definition of $v_2(x, \lambda)$. However, the second term of this expression is asymptotically negligible in the presence of the first. It follows that

$$\text{(11c)} \quad \begin{aligned} v_2(h-, \lambda) &= \frac{1}{2}\left[\frac{\sigma(0)}{\sigma(h-)}\right]^{1/2}\left\{1 + \frac{1}{\lambda}\left[\tau(h) + \frac{\sigma'(0)}{2\sigma(0)}\right] + O\left(\frac{1}{\lambda^2}\right)\right\} e^{\lambda h}, \\ v_2'(h-, \lambda) &= \left\{\lambda - \frac{\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} v_2(h-, \lambda). \end{aligned}$$

With the results (11a), (11b), and (11c) at hand, the evaluation of the right-hand member of the relation (10) is now immediate. It is found thus that

$$\Omega_1(\lambda) - \Omega(\lambda) = \frac{2(A-1)e^{-2h\lambda}}{(A+1)(B+1)},$$

where

$$A = \mu + (\nu - 1)\frac{\sigma'(h-)}{2\lambda\sigma(h-)} + O\left(\frac{1}{\lambda^2}\right),$$

and

$$B = \frac{1}{\lambda}\left[\tau(h) + \frac{\sigma'(0)}{\sigma(0)}\right] + O\left(\frac{1}{\lambda^2}\right),$$

a formula which is more conveniently written according as $\mu \neq 1$ or $\mu = 1$ in the appropriate one of the following forms. That is, if $\mu \neq 1$,

$$\begin{aligned}(12)\qquad \log\left\{\frac{1}{2}[\Omega_1(\lambda) - \Omega(\lambda)]\right\} &= -2h\lambda + \log\left(\frac{\mu-1}{\mu+1}\right) \\ &\quad + \frac{1}{\lambda}\left[\frac{(\nu-1)\sigma'(h-)}{(\mu^2-1)\sigma(h-)} - \frac{\sigma'(0)}{\sigma(0)} - \tau(h)\right] - O\left(\frac{1}{\lambda^2}\right),\end{aligned}$$

and if $\mu = 1$,

$$\begin{aligned}(13)\qquad &\log\left\{\frac{\lambda}{2}[\Omega_1(\lambda) - \Omega(\lambda)]\right\} \\ &\quad = -2h\lambda + \log\left[\frac{(\nu-1)\sigma'(h-)}{4\sigma(h-)}\right] + O\left(\frac{1}{\lambda}\right).\end{aligned}$$

The analysis of these results may now be outlined very simply. The function $\Omega(\lambda)$ is to be regarded as known from the experimentally determined surface potentials. From it the functions $\sigma_1(x)$ and $\Omega_1(\lambda)$ are successively computable. The left-hand members of the relations (12) and (13) are, therefore, at hand. Let the values of (12) be plotted against λ. Then if the resulting graph shows a rectilinear asymptote, it follows from the significance of the respective terms on the right of (12) that the slope and intercept of this asymptote are respectively equal to $-2h$ and to $\log[(\mu-1)/(\mu+1)]$. Hence the graph serves to determine the constants h and μ. An analysis of the distance between the graph and its asymptote would clearly give the value of ν. If, on the other hand, the graph in question shows no asymptote, that fact signifies that $\mu=1$. In that case the left member of (13) is to be plotted. Since ν is now not equal to unity this graph will show an asymptote, the slope and intercept of which determine h and ν.

HARVARD UNIVERSITY

A SERIES OF INVOLUTORIAL CREMONA SPACE TRANSFORMATIONS DEFINED BY A PENCIL OF RULED CUBIC SURFACES*

BY AMOS BLACK

1. *Introduction.* A series of involutorial Cremona transformations of space were defined by Snyder† by means of a correspondence between the surfaces of a pencil of ruled surfaces and the points of a rational curve, called the director curve. The director curve was a part of the basis of the pencil and two of the chief characteristics of the transformation were: one of the principal surfaces was a ruled surface R, all of whose generators were parasitic lines; all the tangent planes of the surfaces of the homaloidal web along a certain curve were fixed, being determined by the surface R.

In this paper we shall define a series of transformations by means of a correspondence between the surfaces of a pencil of ruled cubic surfaces and the points of certain rational curves. The director curve is not part of the basis of the pencil; one of the principal surfaces is a ruled surface R, all of whose generators are parasitic lines; all the surfaces of the homaloidal web have fixed tangent planes along a certain curve, but none of the fixed planes are determined by R.

2. *Definition of the Transformation.* Given a pencil of ruled cubic surfaces $|F_3|$ whose basis curve consists of a double line d

* Presented to the Society, September 13, 1935.

† Virgil Snyder, *On a series of involutorial Cremona transformations of space defined by a pencil of ruled surfaces*, Transactions of this Society, vol. 35 (1933), pp. 341–347.