Hence the bracket symbols form a semi-ring. The commutative law of addition does not hold in general in this semi-ring since

$$[A, \beta] + [B, \gamma] = [AB, \beta + \gamma],$$
$$[B, \gamma] + [A, \beta] = [BA, \gamma + \beta],$$

but $BA \neq AB$. These symbols have a property under addition which might be called quasi-commutativity:

$$[\alpha, \beta] + [\alpha, \beta] + [\gamma, \delta] + [\gamma, \delta] = [\alpha, \beta] + [\gamma, \delta] + [\alpha, \beta] + [\gamma, \delta],$$

for the left-hand member reduces to $[\gamma, \beta+\beta+\delta+\delta]$ and the right to $[\gamma, \beta+\delta+\beta+\delta]$, which are equal since A, B, and C are commutative under addition. It is also easy to see that $MNMN = MMNN$, for M and N are bracket symbols.

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BRANCHED AND FOLDED COVERINGS*

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A simple example of a *branched* covering arises when one sphere is mapped on another so that each point of the first sphere goes into the point of the second which has the same latitude but double the longitude. This is a covering of degree two with simple branching at the north and south poles. As an example of a *folded* covering we take a torus, thought of as a sphere with a handle on one side, and project it radially inward on a smaller concentric sphere. The torus covers the sphere once but with a fold produced by collapse of the handle. The product of this torus-sphere covering with the previous sphere-sphere covering yields a torus-sphere covering of degree two which is both *branched and folded*. Suitable triangulation of the torus and the spheres will turn the above mappings into simplicial mappings in which each simplex maps barycentrically into a simplex of the same dimension. In what follows we make some rudimentary calculations concerning the branching and folding of a simplicial covering of one n-dimensional complex by

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another. We obtain two formulas which connect the Euler characteristics (numbers) of the complexes with those of the subcomplexes about which the branching and folding occur.

Let $K \rightarrow K'$ be a (continuous) simplicial mapping of one simplicial n-complex, K, on another, K'. For the sake of simplicity we assume that K and K' are absolute orientable manifolds* and that each p-simplex s of K goes into a p-simplex s' of K', that is, that no simplex s collapses into one of lower dimension. Later, however, we shall weaken these assumptions in various ways. Let the fundamental n-cycles C, C' of K, K' be oriented so that $C \rightarrow dC'$, where the integer $d \geqq 0$; d is the *degree* of the mapping. Let $C(s)$ denote the part of C which lies on the star of the simplex s, and $C'(s')$ the part of C' which lies on the star of the corresponding s'. Then the integer $d(s)$ defined by $C(s) \rightarrow d(s)C'(s')$ measures the degree of s in the mapping. Normally $d(s) = 1$; if $d(s) \neq 1$ we say that s is *exceptional* in the mapping. If $d(s) > 1$ the *exceptionality* is a branching about s of multiplicity $d(s) - 1$. If $d(s) = 0$, s belongs to the crease of a fold in the covering. If $d(s) < 0$, s belongs to a part of K which gives a negatively sensed layer of a fold; s contributes simply to the layer if $d(s) = -1$, otherwise there is branching about s of multiplicity $|d(s)| - 1$. Our use of the terms *branching* and *fold* is in accord with their customary meaning as applied to coverings, but for the purpose of this paper the descriptions just given may be regarded as definitions of these terms.

Let the integer $e(s) = d(s) - 1$ be taken as a measure of the *exceptionality* of s in the mapping. For each integer $e \times 0$ we form the simplexes of exceptionality $e(s) = e$, if any, into one or more *subcomplexes* $K^{[e]i}$, $(i = 1, 2, \cdots, n_e)$.† In practice the $K^{[e]i}$ will arise as natural connected units about which the branching and folding (and negative stratification) occur. For example, in the torus-sphere covering of degree two described in the opening paragraph the $K^{[1]i}$ = the two branch points at the poles, the $K^{[-1]i}$ = four open segments which constitute the lines of fold, and the $K^{[-2]i}$ = an open region on the inner surface of the

* See Lefschetz, *Topology*, for the terminology.

† Subcomplex is used in the general sense of the author's thesis, *An abstract approach to manifolds*, Annals of Mathematics, (2), vol. 34 (1933), pp. 191–243, as applying to any subset of simplexes on which the (relative) boundary of a (relative) boundary is zero.

handle—a 2-cell bounded by the $K^{[-1]i}$. However, for the purpose of our calculations it does not matter just how the $K^{[e]i}$ are constructed. At worst each $K^{[e]i}$ may consist of an individual simplex.

Summing over all s which map into a given s' we have $\sum d(s)=d$, the degree of the mapping. Therefore, extending the sum over all p-simplexes s, we get $\sum d(s)=d\alpha_p'$, where α_p' is the number of p-simplexes in K'. Hence $\alpha_p+\sum e(s)=d\alpha_p'$. But $\sum e(s)=\sum_e\sum_i e\alpha_p{}^{[e]i}$. Therefore $\alpha_p+\sum_e\sum_i e\alpha_p{}^{[e]i}=d\alpha_p'$, and so

$$X + \sum_e \sum_i eX^{[e]i} = dX',$$

where $X=\sum_p(-1)^p\alpha_p, \cdots$, are the Euler characteristics of K, $\cdots$. This is the first of our two formulas. If the covering has neither branching nor folding the sum on the left side is zero and we have the familiar formula $X=dX'$. If there is branching but no folding only positive values of e occur on the left; we have a generalization of the formula used to characterize a Riemann surface by its branch-points and number of sheets. If there is folding without branching the sum on the left side is zero except for $e=-1, -2$. Values of $e<-2$ go with branching inside negative stratification. In the torus-sphere covering of degree two described in our opening paragraph $X=0$, $X'=2$, $d=2$, $\sum_i(1)X^{[1]i}=2$, $\sum_i(-1)X^{[-1]i}=4$, $\sum_i(-2)X^{[-2]i}=-2$.

In contrast with the preceding work where orientation has played such an essential part we now turn to calculations of an absolute nature, based on $|d(s)|$ rather than on $d(s)$. One justification of this is the fact that the sign of $d(s)$ has no meaning from a local point of view—it was determined by the orientation of C, C' so that $C\rightarrow dC'$ where $d\geqq 0$. We use $\epsilon(s)=|d(s)|-1$ as a measure of the *absolute exceptionality* of s; in terms of $\epsilon(s)$ we form subcomplexes $K^{[\epsilon]j}$, $(j=1, 2, \cdots, n_\epsilon)$, just as we did with $e(s)$ above. Let $\delta(s')$ denote the sum $\sum|d(s)|$ taken over all s which map into a given s'; the values of $\delta(s')$ differ for different s' but are all congruent mod 2 since extra layers occur in pairs. We choose a non-negative number $\delta\leqq$each $\delta(s')$ but congruent to each mod 2, and we set $\delta(s')=\delta+2\lambda(s')$. The non-negative integer $\lambda(s')$ measures the number of pairs of layers there are over s' in excess of the basic number which we have chosen. From the simplexes s' for which $\lambda(s')\geqq\mu>0$ we form subcomplexes $K'^{[\mu]k}$, $(k=1, 2, \cdots, n_\mu)$. Then summing over all p-simplexes s we get $\sum|d(s)| = \delta\alpha_p' + 2\sum_\mu\sum_k\alpha_p'^{[\mu]k}$. Hence $\alpha_p + \sum\epsilon(s) = \delta\alpha_p' + 2\sum_\mu\sum_k\alpha_p'^{[\mu]k}$. But $\sum\epsilon(s) = \sum_\epsilon\sum_j\epsilon\alpha_p^{[\epsilon]j}$. Therefore $\alpha_p + \sum_\epsilon\sum_j\epsilon\alpha_p^{[\epsilon]j} = \delta\alpha_p' + 2\sum_\mu\sum_k\alpha_p'^{[\mu]k}$, and so

$$X + \sum_\epsilon \sum_j \epsilon X^{[\epsilon]j} = \delta X' + 2\sum_\mu \sum_k X'^{[\mu]k}.$$

This is the second of our two formulas. It applies even if K' is not orientable (in which case K may or may not be orientable). If K' is orientable the degree d will probably be taken as the value of δ, but this is not necessary.

We started with the assumption that K, K' were orientable absolute n-manifolds, but the only use we have made of this assumption has been for the n-cycles C, C' and the relative n-cycles $C(s)$, $C'(s')$. All we really need to assume is that K, K' are orientable n-circuits and that the star of each simplex of K' carries an irreducible basis $C'(s') \neq 0$; for the second formula we may dispense with the orientability of K' and leave the orientability of K an open question.

We may also weaken the assumption that in the mapping $K \rightarrow K'$ each simplex s goes into a simplex s' of like dimension. It is sufficient to suppose that K can be divided into p-cell-like* subcomplexes S which map into p-simplexes s'. These S take the place of the simplexes s in the preceding work; a p-cell-like S, like a p-simplex s, has an Euler characteristic $(-1)^p$. The *star* of an S would be the minimal open subcomplex containing S and composed of S's. This extension to more general simplicial coverings has importance for simplicial approximation of continuous coverings.

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* See Tucker, loc. cit. The following is a simple example of a 1-cell-like S mapping into a 1-simplex s'. Let three tetrahedra $ABCD$, $ABDE$, $ABEF$ be collapsed by the mapping $A \rightarrow A'$ and $B, C, D, E, F \rightarrow B'$. The three tetrahedra and all their faces and edges incident with A are mapped into $A'B'$. The aggregate of these simplexes is readily seen to be a 1-cell-like subcomplex ($=$a solid cone with vertex and base removed).