ON THE SUMMABILITY OF FOURIER SERIES

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1. Introduction. It is well known that the Abel method of summability is stronger than the Cesàro methods of any order. An example has been given* to show that there are series which are Abel summable but not Cesàro summable for any order. This series is one for which \( a_n \neq o(n^\alpha) \) for any \( \alpha \), and hence which cannot be \((C, \alpha)\) summable for any \( \alpha \). This series cannot be a Fourier series since for all Fourier series \( a_n = o(1) \). We propose to give an example of the existence of a Fourier series which is Abel summable but not Cesàro summable.

We shall make use of some results of Paley† which show that, if the Fourier series of \( f(x) \),

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]

is \((C, \alpha)\) summable at the point \( x \), then, for \( \beta > \alpha \),

\[
R_\beta(f, t) = \beta \int_0^t \left\{ f(x + \tau) - 2f(x) + f(x - \tau) \right\} (t - \tau)^{\beta-1} d\tau = o(t^\beta), \quad \text{as } t \to 0,
\]

and conversely, if \( R_\alpha(f, t) = o(t^n) \), as \( t \to 0 \), then the series (1) is \((C, \beta)\) summable for every \( \beta > \alpha + 1 \). We shall first show that for every \( n > 1 \) there is a function \( f_n(x) \) such that at \( x = 0 \)

\[
\lim_{t \to 0} \left| \frac{1}{t^j} R_j(f_n, t) \right| = \infty, \quad (j \leq n - 1),
\]

but

\[
R_n(f_n, t) = o(t^n), \quad \text{as } t \to 0.
\]

This implies that the Fourier series of \( f_n(x) \) is \((C, n+2)\) summable at \( x = 0 \) and therefore Abel summable. The function

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is then defined with the $d_n$'s so chosen that the Fourier series of $f(x)$ is Abel summable, but for every $n$

$$R_n(f, t) \neq o(t^n), \quad \text{as } t \to 0.$$ 

This implies, by the theorem of Paley, that the Fourier series of $f(x)$ cannot be $(C, \alpha)$ summable for any $\alpha$.

2. Properties of $f_n(x)$. We suppose for the moment that $n$ is fixed and we let $c = (1 + 1/(n - 1/2))$. We define $a_r = 2^{-c_r}$, $b_r = 2^{r-1} - a_r$; then, if $\nu \geq n$, $b_r > 2^{-i(r+1)}$, so that the intervals $(b_r, 2^{r})$ are non-overlapping for $\nu \geq n$. We define

$$f_n(x) = \begin{cases} 2^r, & b_r \leq |x| \leq b_r + \frac{a_r}{2^n}, \quad (\nu = n, n + 1, \ldots), \\ -f_n \left( x - 2^j \frac{a_r}{2^n} \right), & b_r + 2^j \frac{a_r}{2^n} < |x| \leq b_r + 2^{i+1} \frac{a_r}{2^n}, \\ 0, & \text{elsewhere on } (-\pi, \pi). \end{cases}$$

Then $f_n(x) \in L$ on $(-\pi, \pi)$, for

$$\int_{-\pi}^{\pi} |f_n(x)| \, dx = 2 \sum_{\nu=n}^{\infty} 2^\nu a_\nu = 2 \sum_{\nu=n}^{\infty} 2^{-\nu/(n-1/2)} < \infty.$$

At $x = 0$, $f_n(x+t)+f_n(x-t)-2f_n(x) = 2f_n(t)$. We have

$$\int_{b_r}^{b_r+2(a_r/2^n)} f_n(t) \, dt = \int_{b_r}^{b_r+a_r/2^n} f_n(t) \, dt - \int_{b_r}^{b_r+a_r/2^n} f_n(t) \, dt = 0.$$ 

By the definition of $f_n(x)$,

$$f_n(t) = -f \left( t - 2^j \frac{a_r}{2^n} \right), \quad b_r + 2^j \frac{a_r}{2^n} < t \leq b_r + 2^{i+1} \frac{a_r}{2^n},$$

so that by induction

$$\int_{b_r}^{b_r+2^j(a_r/2^n)} f_n(t) \, dt = 0, \quad (1 \leq j \leq n);$$

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and therefore, if \( b_r + 2i(a_r/2^n) < t \), \( 1 \leq j \leq n-1 \),

\[
R_k(f_n, t) = 2 \int_{b_r + 2i(a_r/2^n)}^{t} f_n(\tau) d\tau.
\]

Hence, if \( b_r + 2i/2(a_r/2^n) < t < b_r + 2i2(a_r/2^n) \), \( 0 \leq j \leq n-2 \),

\[
R_k(f_n, t) = - R_k \left( f_n, t - 2i \cdot 2^{n-1} \cdot \frac{a_r}{2} \right).
\]

Since

\[
R_{k+1}(f_n, t) = (k + 1) \int_0^t R_k(f_n, \tau) d\tau,
\]

we see that in the same way, if \( t > b_r \),

\[
\frac{1}{k+1} R_{k+1}(f_n, t) = \int_{b_r}^t R_k(f_n, \tau) d\tau,
\]

and, for

\[
b_r + 2i \cdot 2^{k+1}(a_r/2^n) < t < b_r + 2i+1 \cdot 2^{k+1}(a_r/2^n), \quad (j+1 \leq n-2),
\]

\[
R_{k+1}(f_n, t) = - R_{k+1} \left( f_n, t - 2i \cdot 2^{k+1} \cdot \frac{a_r}{2^n} \right).
\]

Therefore, for \( k \leq n-1 \),

\[
R_k \left( f_n, b_r + \frac{a_r}{2^n} \right) = 2k2^r \int_0^{a_r/2^n} \left( \frac{a_r}{2^n} - t \right)^{k-1} dt
\]

\[
= 2^{r+1} \left( \frac{a_r}{2^n} \right)^k = o(2^r) \quad \text{as} \quad n \to \infty.
\]

Finally, if \( b_r \leq t < 2^{-r} \),

\[
R_n(f_n, t) = 2n \int_0^t f_n(\tau)(t - \tau)^{n-1} d\tau = O \left( 2^r \int_{b_r}^t (t - \tau)^{n-1} d\tau \right)
\]

\[
= O(2^r a_r^{n}) = O(2^r 2^{-nr/(n-1/2)} 2^{-nr}) = o(2^{-nr}) = o(t) \quad \text{as} \quad t \to 0.
\]

Therefore the function \( f_n(x) \) has the properties (2) and (3).

3. A Function whose Fourier Series is not Summable \((C, \alpha)\).

As we have already mentioned, the Fourier series of \( f_n(x) \) will be Abel summable at \( x = 0 \). Therefore,
\[ A_n = \begin{cases} 1 \text{. u.b.} & \text{if } 0 \leq r < 1 \\
\frac{1}{2\pi} \int_0^\pi \{ f_n(x+t) + f_n(x-t) - 2f_n(x) \} \frac{1-r^2}{1-2 \cos t + r^2} dt & \text{if } r \geq 1 
\end{cases} \]

will exist. We may define two sequences \( \{d_n\} \) and \( \{t_n\} \) simultaneously by induction so that

\[ (4) \quad d_n \leq \min \left( \frac{1}{2^n A_n}, \frac{1}{2^n}, \frac{1}{2^n \int_{-\pi}^\pi |f_n(t)| \, dt} \right), \]

\[ (5) \quad d_n \leq \frac{1}{2^n} \min_{r \leq n-2} \left( \frac{1}{R_n^{-1}(f_n, t_{n+1})} \right), \]

\[ (6) \quad \left| t_n^{-(n-1)} R_{n-1}(f_n, t_n) \right| > \frac{n}{d_n}, \]

\[ (7) \quad \left| t_n^{-(n-1)} R_{n-1}(f_n, t_n) \right| < \frac{1}{n}, \quad (\nu \leq n-1). \]

It is clear that \( d_n \) can be chosen so as to satisfy (4) and (5). It is possible to choose \( t_n \) satisfying (6) and (7), since

\[ \lim_{t \to 0} \left| t^{-(n-1)} R_{n-1}(f_n, t) \right| = \infty, \]

and

\[ t^{-\mu} R_{\mu}(f_n, t) = o(1) \quad \text{as } t \to 0, \quad \text{for } \mu \geq n. \]

The function

\[ f(x) = \sum_{n=2}^{\infty} d_n f_n(x) \]

is integrable, for

\[ \int_{-\pi}^{\pi} |f(x)| \, dx \leq \sum_{n=2}^{\infty} d_n \int_{-\pi}^{\pi} |f_n(x)| \, dx \leq \sum_{n=2}^{\infty} 2^{-n}. \]

The Fourier series of \( f(x) \) is Abel summable, since

\[ A(f, r) = \sum_{n=1}^{\infty} d_n A(f_n, r), \]

and \( d_n A(f_n, r) \leq 1/2^n \), and \( A(f_n, r) \to 0 \) as \( r \to 1 \), which implies that \( A(f, r) \to 0 \) as \( r \to 1 \).
We shall show that, for every \( n \), \( R_n(f, t) \neq o(t^n) \), as \( t \to 0 \). Let us suppose that, for some \( n \), \( R_n(f, t) = o(t^n) \), as \( t \to 0 \); then, since

\[
R_{n+1}(f, t) = (n+1) \int_0^t R_n(f, \tau) d\tau,
\]

there would be a constant \( K \) such that for all \( t \) and \( m \geq n \) we would have

\[
(8) \quad | R_m(f, t) | \leq K t^m.
\]

We shall show that for every \( n \)

\[
| t_n^{-(n-1)} R_{n-1}(f_n, t_n) | > n + o(1), \quad \text{as} \quad n \to \infty,
\]

which contradicts (8). We have

\[
t_n^{-(n-1)} R_{n-1}(f_n, t_n) = \sum_{r=2}^{\infty} d_r t_n^{-(r-1)} R_{n-1}(f_r, t_n)
\]

\[
= \sum_{r=2}^{n-1} d_r t_n^{-(r-1)} R_{n-1}(f_r, t_n) + d_n t_n^{-(n-1)} R_{n-1}(f_n, t_n)
\]

\[
+ \sum_{r=n+1}^{\infty} d_r t_n^{-(r-1)} R_{n-1}(f_r, t_n).
\]

By (7),

\[
\left| \sum_{r=2}^{n-1} d_r t_n^{-(r-1)} R_{n-1}(f_r, t_n) \right| < \frac{1}{n} \sum_{r=2}^{n-1} \left| d_r \right| = o(1), \quad \text{as} \quad n \to \infty,
\]

and, by (5),

\[
\left| \sum_{r=n+1}^{\infty} d_r t_n^{-(r-1)} R_{n-1}(f_r, t_n) \right| \leq \sum_{r=n+1}^{\infty} 2^{-r} = o(1), \quad \text{as} \quad n \to \infty,
\]

so that, by (6),

\[
| t_n^{-(n-1)} R_{n-1}(f_n, t_n) | = \left| d_n t_n^{-(n-1)} R_{n-1}(f_n, t_n) \right| + o(1)
\]

\[
> n + o(1), \quad \text{as} \quad n \to \infty.
\]

Therefore by the theorem of Paley the Fourier series of \( f(x) \) cannot be \((C, \alpha)\) summable for any \( \alpha \).