GENERALIZED CONVEX FUNCTIONS

BY E. F. BECKENBACH

1. *Introduction.* We shall be concerned in this paper with real finite functions $f(x)$ defined in an interval $a<x<b$. A function $f(x)$ is said to be *convex** in (a, b) provided, for an arbitrary subinterval (x_1, x_2) interior to (a, b), the curve $y=f(x)$ lies nowhere above the line segment joining its end points; that is, provided, for arbitrary x_1, x_2, x, with $a<x_1<x<x_2<b$,

$$f(x) \leqq F_{12}(x),$$

where $F_{12}(x)$ is the function of the form

$$F_{12}(x) = \alpha x + \beta$$

satisfying

$$F_{12}(x_1) = f(x_1), \qquad F_{12}(x_2) = f(x_2).$$

Several generalizations of the notion of a convex function to other classes of functions of one variable have found their way into the literature.† In what follows we discuss the notions which seem to underlie these classes of functions.

Let $F(x; \alpha, \beta)$ be a (two-parameter) family of real finite functions defined for $a<x<b$ and satisfying the following conditions:

(1) each $F(x; \alpha, \beta)$ is a continuous function of x;

(2) there is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a<x_1<x_2<b$, takes on arbitrary values y_1, y_2.

For example, such a simple $F(x; \alpha, \beta)$ as $x^2+\alpha x+\beta$ is not in-

* J. L. W. V. Jensen, *Sur les fonctions convexes et les inégalités entre les valeurs moyennes*, Acta Mathematica, vol. 30 (1906), pp. 175–193.

† See E. Phragmén and E. Lindelöf, *Sur une extension d'un principe classique de l'analyse*, Acta Mathematica, vol. 31 (1908), pp. 381–406; G. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Mathematische Zeitschrift, vol. 29 (1929), pp. 549–640; B. Jessen, *Über die Verallgemeinerungen des arithmetischen Mittels*, Acta Szeged, vol. 5 (1931), pp. 108–116; G. Valiron, *Fonctions convexes et fonctions entières*, Bulletin de la Société Mathématique de France, vol. 60 (1932), pp. 278–287.

cluded in any of the above mentioned generalizations. Other examples are $\alpha x+\beta$; $Ax^3+Bx^2+\alpha x+\beta$, where A and B are fixed constants; $\alpha e^{\rho x}+\beta e^{-\rho x}$, where ρ is a fixed constant; and, with $b-a\leqq\pi/\rho$, $\alpha \sin \rho x+\beta \cos \rho x$. Still another example of the family $F(x;\alpha,\beta)$ is the set of images of all non-vertical straight lines, under a one-to-one continuous transformation of the domain $a<x<b$ of the plane into itself in such a way that every vertical line is transformed into itself.

Members of the above family shall be denoted simply by $F(x)$, not $F(x;\alpha,\beta)$, individual members being distinguished by subscripts. In particular, $F_{ij}(x)$ shall denote the member satisfying

$$F_{ij}(x_i) = f(x_i), \qquad F_{ij}(x_j) = f(x_j), \qquad (a < x_i < x_j < b).$$

DEFINITION 1. The function $f(x)$ shall be called a *sub-$F(x;\alpha,\beta)$ function*, or simply a *sub-$F(x)$ function*, provided

$$f(x) \leqq F_{12}(x) \tag{3}$$

for all x_1, x_2, x, with $a<x_1<x<x_2<b$.

DEFINITION 2. *Super-$F(x)$ functions* are defined exactly as are sub-$F(x)$ functions, excepting that the sign of inequality is reversed; the analysis is the same, *mutatis mutandis*.

2. *The Family* $F(x;\alpha,\beta)$. We shall prove the following theorem.

THEOREM 1. *Let $a<x_0<b$, and let $F_r(x)$, $F_s(x)$ be two members of the family satisfying* (1) *and* (2) *such that*

$$F_r(x_0) = F_s(x_0), \tag{4}$$

$$F_r(x) \not\equiv F_s(x), \qquad (a < x < b); \tag{5}$$

then $F_r(x)>F_s(x)$ for all x in (a, b) on one side of x_0, while $F_r(x)<F_s(x)$ for all x in (a, b) on the other side of x_0.

PROOF. By (2), (4), and (5), $F_r(x)\neq F_s(x)$, $(a<x<b)$, except at x_0; consequently, by (1), on either side of x_0, one of $F_r(x)$, $F_s(x)$ is greater than the other. Suppose it could be the same one, say $F_s(x)$, which is greater on each side; we shall show this is impossible.

Let $a<x_1<x_0<x_2<b$, and consider $F_t(x)$, determined by

$$F_t(x_1) = F_s(x_1), \qquad F_t(x_2) = F_r(x_2).$$

Then $F_t(x_2) < F_s(x_2)$, so that $F_t(x) < F_s(x)$, $(x_1 < x < b)$; in particular,

$$F_t(x_0) < F_s(x_0). \tag{6}$$

Similarly,

$$F_t(x_0) > F_r(x_0). \tag{7}$$

Now (6) and (7) contradict (4), establishing the theorem.

COROLLARY 1. *If $a < x_1 < x_2 < b$, and if $F_r(x_1) < F_s(x_1)$, $F_r(x_2) < F_s(x_2)$, then $F_r(x) < F_s(x)$ for $x_1 \leqq x \leqq x_2$; but if $F_r(x_1) < F_s(x_1)$, $F_r(x_2) > F_s(x_2)$, then $F_r(x) < F_s(x)$ for $a < x \leqq x_1$, and $F_r(x) > F_s(x)$ for $x_2 \leqq x < b$.*

THEOREM 2. *Let the points (x_n, y_n), (x_n', y_n'), $(n = 0, 1, 2, \cdots)$, satisfy $a < x_n < x_n' < b$, and*

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0), \qquad \lim_{n\to\infty} (x_n', y_n') = (x_0', y_0'),$$

and let $F_n(x)$ be determined by

$$F_n(x_n) = y_n, \qquad F_n(x_n') = y_n'; \tag{8}$$

then

$$\lim_{n\to\infty} F_n(x) = F_0(x), \qquad (a < x < b), \tag{9}$$

uniformly in any closed subinterval of $a < x < b$.

PROOF. For a given $\epsilon > 0$, and for an arbitrary $\bar{x}$ in (a, b), with $\bar{x} \neq x_0$, $\bar{x} \neq x_0'$, let the functions $F_p(x), \cdots, F_s(x)$ be determined by

(10) $F_p(x_0) = F_0(x_0)$, (11) $F_p(\bar{x}) = F_0(\bar{x}) + \epsilon$,

(12) $F_q(x_0) = F_0(x_0)$, (13) $F_q(\bar{x}) = F_0(\bar{x}) - \epsilon$,

(14) $F_r(x_0') = F_p(x_0')$, (15) $F_r(\bar{x}) = F_p(\bar{x}) + \epsilon = F_0(\bar{x}) + 2\epsilon$,

(16) $F_s(x_0') = F_q(x_0')$, (17) $F_s(\bar{x}) = F_q(\bar{x}) - \epsilon = F_0(\bar{x}) - 2\epsilon$.

Let δ denote the least of the four (minimum) distances from the curves $y = F_r(x)$ and $y = F_s(x)$ to the points (x_0, y_0) and (x_0', y_0'). It is easily verified, by Theorem 1 and (10)-(17), that $\delta > 0$. Let

$$\delta^* = \min(\delta, |\bar{x} - \bar{x}_0|, |\bar{x} - x_0'|); \tag{18}$$

then $\delta^* > 0$.

Now let $n_0 = n_0(\bar{x})$ be so large that

$$\begin{gathered} \{(x_n - x_0)^2 + (y_n - y_0)^2\}^{1/2} < \delta^*, \\ \{(x_n' - x_0')^2 + (y_n' - y_0')^2\}^{1/2} < \delta^*, \qquad (n \geqq n_0). \end{gathered} \tag{19}$$

We shall prove that

$$|F_n(\bar{x}) - F_0(\bar{x})| < 2\epsilon, \qquad (n \geqq n_0). \tag{20}$$

Three cases arise, as follows.

CASE 1. $x_0 < \bar{x} < x_0'$. In this case, by (10), (11), and Theorem 1, $F_p(x_0') > F_0(x_0')$; consequently, by (10), (14), (15), and Theorem 1, $F_r(x_0) > F_0(x_0)$, $F_r(x_0') > F_0(x_0')$. Similarly, $F_s(x_0) < F_0(x_0)$, $F_s(x_0') < F_0(x_0')$. By (1) and (19), then, for $n \geqq n_0$,

$$\begin{gathered} F_s(x_n) < y_n = F_n(x_n) < F_r(x_n), \\ F_s(x_n') < y_n' = F_n(x_n') < F_r(x_n'), \end{gathered}$$

whence it follows from Corollary 1 that

$$F_s(x) < F_n(x) < F_r(x), \qquad (x_n < x < x_n', n \geqq n_0). \tag{21, A}$$

In particular,

$$F_s(\bar{x}) < F_n(\bar{x}) < F_r(\bar{x}), \qquad (n \geqq n_0). \tag{22}$$

Now (20) follows from (15), (17) and (22).

CASE 2. $x_0' < \bar{x} < b$. The above analysis gives this time

$$F_r(x_0) < F_0(x_0) < F_s(x_0), \qquad F_r(x_0') > F_0(x_0') > F_s(x_0'),$$

and therefore, for $n \geqq n_0$,

$$\begin{gathered} F_s(x_n) > y_n = F_n(x_n) > F_r(x_n), \\ F_s(x_n') < y_n' = F_n(x_n') < F_r(x_n'), \end{gathered}$$

whence it follows from Corollary 1 that

$$F_s(x) < F_n(x) < F_r(x), \qquad x_n' < x < b, \qquad (n \geqq n_0). \tag{21, B}$$

In particular, (22), and therefore (20), hold in this case.

CASE 3. $a < \bar{x} < x_0$. The same analysis gives now

$$F_r(x_0) > F_0(x_0) > F_s(x_0), \qquad F_r(x_0') < F_0(x_0') < F_s(x_0'),$$

and therefore, for $n \geqq n_0$,

$$F_s(x_n) < y_n = F_n(x_n) < F_r(x_n),$$
$$F_s(x_n') > y_n' = F_n(x_n') > F_r(x_n'),$$

whence

$$\text{(21, C)} \qquad F_s(x) < F_n(x) < F_r(x), \qquad (a < x < x_n;\ n \geqq n_0),$$

so that again (22) and (20) hold.

For each x in (a, b), except for x_0, x_0', (9) follows from (20). Let x_t, x_t', with $x_t < x_t'$, be any other two points in (a, b). Then the sequences

$$\text{(23)} \qquad (x_t, F_n(x_t)), \quad (x_t', F_n(x_t')), \qquad (n = 1, 2, 3, \cdots),$$

satisfy, by the above analysis,

$$\text{(24)} \qquad \begin{aligned} \lim_{n\to\infty} (x_t, F_n(x_t)) &= (x_t, F_0(x_t)), \\ \lim_{n\to\infty} (x_t', F_n(x_t')) &= (x_t', F_0(x_t')). \end{aligned}$$

Now since (24) holds, the above analysis can be applied to the sequences (23), giving (20) this time for $\bar{x} \neq x_t$, $\bar{x} = x_t'$, in particular for $\bar{x} = x_0$ and for $\bar{x} = x_0'$. Therefore (20) and (9) hold throughout (a, b).

In each of the above three cases, by (1), there is an interval $\bar{x} - \eta < x < \bar{x} + \eta$, where $\eta = \eta(\bar{x}) > 0$, in which

$$F_s(x) > F_s(\bar{x}) - \epsilon, \quad F_r(x) < F_r(\bar{x}) + \epsilon, \quad |F_0(x) - F_0(\bar{x})| < \epsilon,$$

and therefore, by (15), (17), and (21, A, B, C), in which

$$\text{(25)} \qquad \begin{aligned} &|F_n(x) - F_0(x)| < 4\epsilon, \\ &(\bar{x} - \eta(\bar{x}) < x < \bar{x} + \eta(\bar{x});\ n \geqq n_0(\bar{x})). \end{aligned}$$

The same discussion holds, according to the preceding paragraph, for intervals about x_0, x_0'. Then (25) holds for arbitrary $\bar{x}$, with $a < \bar{x} < b$, whence the uniformity of (9) for any closed subinterval of $a < x < b$ follows from a simple application of the familiar Heine-Borel Theorem.

Corollary 2. *Any subset of the family $F(x;\alpha,\beta)$ is compact, provided the ordinates are bounded for two distinct abscissas.*

PROOF. One readily sets up sequences satisfying the conditions of Theorem 2.

3. *Some Properties of Sub-$F(x;\alpha,\beta)$ Functions.*

THEOREM 3. *If $f(x)$ is a sub-$F(x)$ function, then for any x_1, x_2, with $a<x_1<x_2<b$, the inequality $f(x)\geqq F_{12}(x)$ holds for $a<x<x_1$, and for $x_2<x<b$.*

PROOF. Fix x_3, with $x_2<x_3<b$. Since

$$F_{13}(x_1) = F_{12}(x_1) = f(x_1), \qquad F_{13}(x_2) \geqq F_{12}(x_2) = f(x_2),$$

it follows from Theorem 1 that

$$f(x_3) = F_{13}(x_3) \geqq F_{12}(x_3).$$

A similar proof holds for $a<x_3<x_1$.

THEOREM 4. *If $f(x)$ is a sub-$F(x)$ function, and if, for some x_1, x_2, x_3, with $a<x_1<x_3<x_2<b$,*

$$f(x_3) = F_{12}(x_3), \tag{26}$$

then $f(x)\equiv F_{12}(x)$, $(x_1\leqq x\leqq x_2)$.

PROOF. By (2) and (26),

$$F_{12}(x) \equiv F_{13}(x) \equiv F_{32}(x); \tag{27}$$

consequently, by Theorem 3 and (27),

$$F_{12}(x) \equiv F_{13}(x) \leqq f(x), \qquad (x_3 \leqq x < b), \tag{28}$$

$$F_{12}(x) \equiv F_{32}(x) \leqq f(x), \qquad (a < x \leqq x_3). \tag{29}$$

But according to (3),

$$F_{12}(x) \geqq f(x), \qquad (x_1 \leqq x \leqq x_2). \tag{30}$$

The theorem follows from inequalities (28), (29), and (30).

THEOREM 5. *If $f(x)$ is a sub-$F(x)$ function, and if $a<x_3\leqq x_1<x_2\leqq x_4<b$, then*

$$F_{34}(x) \geqq F_{12}(x), \qquad (x_3 \leqq x \leqq x_4). \tag{31}$$

PROOF. By Theorem 3,

$$F_{34}(x_3) = f(x_3) \geqq F_{12}(x_3), \qquad F_{34}(x_4) = f(x_4) \geqq F_{12}(x_4),$$

whence, by Theorem 1 and its corollary, (31) follows. Incidentally, by (2) and Theorem 1, the sign of equality in (31) holds nowhere in $x_3 \leqq x \leqq x_4$, except at most at one of the end points, unless it holds identically.

THEOREM 6. *If $f(x)$ is a sub-$F(x)$ function, then $f(x)$ is continuous.*

PROOF. We shall show that $f(x)$ is continuous at an arbitrary x_0, with $a < x_0 < b$. Let $a < x_1 < x_0 < x_2 < b$. Then, by Definition 1 and Theorem 3, for $x_1 < x_0 - h < x_0 + h < x_2$,

$$F_{10}(x_0 - h) \geqq f(x_0 - h) \geqq F_{02}(x_0 - h),$$
$$F_{10}(x_0 + h) \leqq f(x_0 + h) \leqq F_{02}(x_0 + h).$$

Let $h \rightarrow 0$; the theorem follows from (1).

REMARK. *Though a convex function necessarily possesses a derivative almost everywhere, not all sub-$F(x)$ functions possess this property.*

PROOF. Let $\phi(x)$ be a nowhere differentiable function; the family

$$F(x; \alpha, \beta) \equiv \phi(x) + \alpha x + \beta$$

satisfies conditions (1) and (2). Now any particular member of this family is itself a sub-$F(x)$ function, but is nowhere differentiable.

4. *Characterization of Sub-$F(x; \alpha, \beta)$ Functions.*

THEOREM 7. *A necessary and sufficient condition that a continuous function $f(x)$ be a sub-$F(x)$ function is that for all x_0, with $a < x_0 < b$, and for all $\delta > 0$, there exist a positive $h = h(x_0, \delta) < \delta$, with*

$$(32) \qquad a < x_1 = x_0 - h < x_0 < x_0 + h = x_2 < b,$$

such that

$$(33) \qquad f(x_0) \leqq F_{12}(x_0).$$

NECESSITY. If $f(x)$ is a sub-$F(x)$ function, then for all h such that (32) is satisfied, (33) holds as a special case of (3).

SUFFICIENCY. If $f(x)$ is not a sub-$F(x)$ function, then by (3) there exist x_3, x_4, x_5, with $a<x_3<x_5<x_4<b$, such that

$$f(x_5) > F_{34}(x_5). \tag{34}$$

By (34) and the continuity of $f(x)$ and of $F_{34}(x)$, there exist x_1, x_2, with $x_3 \leqq x_1 < x_5 < x_2 \leqq x_4$, such that

$$f(x) \geqq F_{34}(x), \qquad (x_1 \leqq x \leqq x_2), \tag{35}$$

the sign of equality holding at the end points but not elsewhere; then $F_{34}(x) \equiv F_{12}(x)$, so that (35) can be written as

$$f(x) > F_{12}(x), \qquad (x_1 < x < x_2). \tag{36}$$

Fix x_6, with $a<x_6<x_1$, and let $F_k(x)$ satisfy

$$F_k(x_6) = F_{12}(x_6), \qquad F_k(x_2) = F_{12}(x_2) + k = f(x_2) + k;$$

in particular,

$$F_0(x) \equiv F_{12}(x). \tag{37}$$

Let k increase continuously from $k=0$; then, for any $x>x_6$ in (a, b), by Theorems 1 and 2, $F_k(x)$ increases continuously. Further, it is easy to show, by Theorem 1 and the Heine-Borel Theorem, that for k sufficiently large,

$$f(x) < F_k(x), \qquad (x_1 \leqq x \leqq x_2). \tag{38}$$

By (36), (37), and (38), there is some largest $k_0>0$ for which

$$f(x) = F_{k_0}(x)$$

has a solution in $x_1 \leqq x \leqq x_2$. Let x_7 be the largest value for which

$$f(x_7) = F_{k_0}(x_7), \qquad (x_1 \leqq x_7 \leqq x_2). \tag{39}$$

Since $k_0>0$, we have $x_1<x_7<x_2$.

Suppose (33) could be satisfied at x_7 for the above function $f(x)$, which is not a sub-$F(x)$ function, and for some arbitrarily small positive h; we shall obtain a contradiction. Let h, with $x_1<x_8=x_7-h<x_{7+h}=x_9<x_2$, be a value for which (33) is satisfied:

$$f(x_7) \leqq F_{89}(x_7). \tag{40}$$

By the choice of k_0,

$$F_{89}(x_8) = f(x_8) \leqq F_{k_0}(x_8),$$
$$F_{89}(x_9) = f(x_9) < F_{k_0}(x_9),$$

so that, by Theorem 1 and its corollary,

$$F_{89}(x) < F_{k_0}(x), \qquad (x_8 < x < b);$$

in particular,

$$F_{89}(x_7) < F_{k_0}(x_7). \tag{41}$$

Now (41) contradicts (39) and (40).

THE RICE INSTITUTE

SUFFICIENT CONDITIONS FOR A NON-REGULAR PROBLEM IN THE CALCULUS OF VARIATIONS*

G. M. EWING

1. *Introduction*. Given $J=\int_{x_1}^{x_2}f(x,y,y')dx$, it is well known that a minimizing curve satisfies the necessary conditions of Euler, Weierstrass, and Legendre, which we shall designate as I, II, and III,† respectively. If further, $f_{y'y'}(x, y, y')\neq 0$ on the minimizing curve, the Jacobi condition IV is necessary, while the stronger set of conditions I, II_b', III′, and IV′‡ are sufficient for a strong relative minimum.

The purpose of this study is to obtain a set of sufficient conditions for a curve without corners along which $f_{y'y'}$ may have zeros. Since the classical theory gives only the necessary conditions I, II, and III, we wish to obtain a Jacobi condition; and with this in view, introduce the integral

$$L \equiv \int_{x_1}^{x_2}\phi(x, y, y')dx, \ \phi(x, y, y') \equiv f(x, y, y') + k^2[y' - e'(x)]^2, \qquad (x_1 \leqq x \leqq x_2,\ k \lessgtr 0),$$

by means of which we find a necessary condition that we shall call IV_L'. Suitably strengthened, this becomes IV_{Lb}' and the set of conditions I, II_b, III_b, and IV_{Lb}' are found sufficient for an improper strong relative minimum.

---

* Presented to the Society, November 27, 1936.

† G. A. Bliss, *Calculus of Variations*, 1925, pp. 130–132.

‡ Bliss, loc. cit., pp. 134–135.