$$F_{89}(x_8) = f(x_8) \leqq F_{k_0}(x_8),$$
$$F_{89}(x_9) = f(x_9) < F_{k_0}(x_9),$$

so that, by Theorem 1 and its corollary,

$$F_{89}(x) < F_{k_0}(x), \qquad (x_8 < x < b);$$

in particular,

$$F_{89}(x_7) < F_{k_0}(x_7). \tag{41}$$

Now (41) contradicts (39) and (40).

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SUFFICIENT CONDITIONS FOR A NON-REGULAR PROBLEM IN THE CALCULUS OF VARIATIONS*

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1. *Introduction*. Given $J = \int_{x_1}^{x_2} f(x, y, y')dx$, it is well known that a minimizing curve satisfies the necessary conditions of Euler, Weierstrass, and Legendre, which we shall designate as I, II, and III,† respectively. If further, $f_{y'y'}(x, y, y') \neq 0$ on the minimizing curve, the Jacobi condition IV is necessary, while the stronger set of conditions I, II_b', III′, and IV′‡ are sufficient for a strong relative minimum.

The purpose of this study is to obtain a set of sufficient conditions for a curve without corners along which $f_{y'y'}$ may have zeros. Since the classical theory gives only the necessary conditions I, II, and III, we wish to obtain a Jacobi condition; and with this in view, introduce the integral

$$L \equiv \int_{x_1}^{x_2} \phi(x, y, y')dx, \; \phi(x, y, y') \equiv f(x, y, y') + k^2[y' - e'(x)]^2, \qquad (x_1 \leqq x \leqq x_2,\ k \lessgtr 0),$$

by means of which we find a necessary condition that we shall call IV_L'. Suitably strengthened, this becomes IV_{Lb}' and the set of conditions I, II_b, III_b, and IV_{Lb}' are found sufficient for an improper strong relative minimum.

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† G. A. Bliss, *Calculus of Variations*, 1925, pp. 130–132.

‡ Bliss, loc. cit., pp. 134–135.

It appears likely that analogous results can be obtained for other problems in the Calculus of Variations and I hope to discuss some of these at a later time.

2. *A Jacobi Necessary Condition.* If E: $y=e(x)$ furnishes at least an improper strong relative minimum for J, it furnishes a proper strong relative minimum for L. Furthermore, if E minimizes J it satisfies III for J. This implies that it satisfies III′ for L, since $\phi_{y'y'}=f_{y'y'}+2k^2$; and the classical treatment then shows that it must satisfy IV for L.

If E satisfies IV (or IV′) for every L, we shall say that it satisfies the condition IV_L (or IV'_L, respectively) for J. Clearly IV_L is necessary. We now show that the same is true of IV'_L.

We write the parameter in L in the form $k^2=(a^2+\alpha)/2$, $a\neq 0$, $\alpha>-a^2$, and consider the Jacobi differential equations*

$$qu''+ru'+su=0, \tag{1}$$

$$(q+a^2+\alpha)u''+ru'+su=0, \tag{2}$$

for J and L, where $q=f_{y'y'}[x, e(x), e'(x)]\geqq 0$ in the closed interval $[x_1, x_2]$ from III, and r and s are other known functions of x. Since q may vanish in $[x_1, x_2]$, the usual existence theorems can not be applied to (1). They do apply to (2), however, the general solution of which for $\alpha=0$ is $u=c_1u_1(x)+c_2u_2(x)$, where the u's constitute a fundamental system and are of class C''† in $[x_1, x_2]$. $\Delta(x, x_1)=\pm u_2(x_1)u_1(x)\mp u_1(x_1)u_2(x)$ is a particular solution vanishing at $x=x_1$. By hypothesis, E: $y=e(x)$ is a minimizing curve satisfying IV_L so that, by proper choice of signs, $\Delta(x, x_1)$ is positive in the interval $x_1<x<x_2$.

For every admissible α (that is, $\alpha>-a^2$) there exists a solution $\Delta(x, x_1, \alpha)$ of (2) vanishing at $x=x_1$ and such that $\Delta'(x_1, x_1, \alpha)=\Delta'(x_1, x_1)$, where $\Delta''(x, x_1, \alpha)$ is continuous in x and of class C' in α.‡

We next study the related equation

$$(q+a^2)u''+ru'+su=-\alpha\Delta''(x, x_1, \alpha), \tag{3}$$

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* Oskar Bolza, *Vorlesungen über Variationsrechnung*, 1933, p. 60.

† That is, they have continuous second derivatives. Bolza, loc. cit., p. 14.

‡ Replace (2) by the system $u'=v$ and $(q+a^2+\alpha)v'+rv+su=0$, and apply the existence theorem given by Bolza, loc. cit., p. 187.

whose general solution can, by the method of variation of parameters, be expressed in the form

$$u = c_1 u_1(x) + c_2 u_2(x) + \alpha A(x, \alpha), \tag{4}$$

where

$$A(x, \alpha) = u_1(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha) u_2(x) dx}{(q + a^2) D(x)} - u_2(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha) u_1(x) dx}{(q + a^2) D(x)},$$

$$D(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \neq 0 \text{ in the closed interval } [x_1, x_2].*$$

$\Delta(x, x_1, \alpha)$, as a particular solution of (3), can be represented in the form (4); and, since it vanishes for $x = x_1$, we obtain

$$\Delta(x, x_1, \alpha) = \lambda \Delta(x, x_1) + \alpha A(x, \alpha), \tag{5}$$

where in general λ is a function of α. Clearly $\lambda(0) = 1$.

E satisfies IV_L by hypothesis. If it fails to satisfy IV' for the L corresponding to $\alpha = 0$, we have

$$\Delta(x_2, x_1, 0) = \lambda(0) \Delta(x_2, x_1) = \Delta(x_2, x_1) = 0,$$

while, if a second $\alpha \neq 0$ has the same property, we have

$$\Delta(x_2, x_1, \alpha) = \lambda(\alpha) \Delta(x_2, x_1) + \alpha A(x_2, \alpha) = \alpha A(x_2, \alpha) = 0.$$

This requires

$$A(x_2, \alpha) = 0. \tag{6}$$

But

$$\begin{aligned} A(x_2, \alpha) &= \int_{x_2}^{x_1} \frac{\Delta''(x, x_1, \alpha) \Delta(x, x_2)}{(q + a^2) D(x)} dx \\ &= \frac{\Delta(\bar{x}, x_2)}{(\bar{q} + a^2) D(\bar{x})} \int_{x_2}^{x_1} \Delta''(x, x_1, \alpha) dx \\ &= \frac{\Delta(\bar{x}, x_2) [\Delta'(x_1, x_1, \alpha) - \Delta'(x_2, x_1, \alpha)]}{(\bar{q} + a^2) D(\bar{x})}, \\ & \qquad (x_1 < \bar{x} < x_2; \ \bar{q} = q(\bar{x})), \end{aligned}$$

* Bolza, loc. cit., p. 75.

where $\Delta(x, x_2)$ is written for $u_2(x_2)u_1(x)-u_1(x_2)u_2(x)$. This fraction can not vanish, the first factor in the numerator being different from zero by IV_L, the second factor being the difference between two terms of opposite sign. Thus (6) is false; and there is at most one L, namely the one for which $\alpha=0$, for which E fails to satisfy IV′.

If $\Delta(x_2, x_1, 0)=0$, we have $\Delta(x_2, x_1, \alpha)=\alpha A(x_2, \alpha)$ from (5). Furthermore $\Delta(x_2, x_1, \alpha)$ must then have a minimum of zero for $\alpha=0$;* so that its derivative, which is $A(x_2, 0)$, must vanish. This is a special case of (6), which has been proved to be false, so that IV_L' is a necessary condition.

3. *Sufficient Conditions for a Minimum for L.* We assume an arc E: $y=e(x)$ satisfying the necessary conditions I, II, III, and IV_L' for J. If II is strengthened to II_b, we can show that this arc satisfies the classical sufficient conditions for L.

Comparing the Euler equations, we see that if E satisfies I for J it does the same for L. The E-functions† for the two problems are related by the equation

$$E_L(x, y, y', Y') \equiv E_J(x, y, y', Y') + k^2(y'-Y')^2,$$

so that II_b for J implies II_b' for L. We have seen in §2 that III for J implies III′ for L and the condition IV_L' requires IV′ for L as a matter of definition.

Hence E furnishes a proper strong minimum to L relative to a certain (x, y) region R, which in general depends on k.‡

4. *Sufficient Conditions for an Improper Strong Relative Minimum for J.* We must find how to strengthen our conditions so as to insure a field§ F which is independent of k. To that end we replace III by III_b and consider the line Λ: $x=x_1$, $y=n\lambda-y_1$, together with a slope function $P(\lambda)\equiv m\lambda+e'(x_1)$. The extremals for L are $y=y(x, a, b, \alpha)$, and the equations

* $\Delta(x_2, x_1, \alpha)>0$ for $\alpha\neq 0$ by IV_L and the choice of signs preceding equation (3).

† This is the only direct reference to the E-function. There need be no confusion with our notation for the curve E: $y=e(x)$.

‡ If E satisfies III′ and IV, but not IV′, for J, R reduces to the curve E as k approaches zero.

§ Bliss, loc. cit., pp. 132–33.

$$n\lambda + y_1 - y(x_1, a, b, \alpha) = 0,$$

$$m\lambda + e'(x_1) - y'(x_1, a, b, \alpha) = 0,$$

define $a = a(\lambda, \alpha) = \bar{a}(y, \alpha)$, and $b = b(\lambda, \alpha) = \bar{b}(y, \alpha)$* for any admissible α and for every y for which (x_1, y) is in the region where III_b holds. These implicit functions are of at least class C' in their respective variables. We thus have a family of extremals of parameter λ for each admissible α,

$$y = \phi(x, \lambda, \alpha) \equiv y[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha],$$

intersecting Λ and including E for $\lambda = 0$. We wish this family to furnish a field.

If there exists an x, $(x_1 < x \leqq x_2)$, such that $\phi(x, \lambda_1, \alpha) - \phi(x, \lambda_2, \alpha) = 0$, there is a $\bar{\lambda}$, $(\lambda_1 < \bar{\lambda} < \lambda_2)$, such that

$$\phi_\lambda(x, \bar{\lambda}, \alpha) = y_a \frac{\partial a}{\partial \lambda} + y_b \frac{\partial b}{\partial \lambda} \Big|_{\lambda = \bar{\lambda}} = 0.$$

This can be expressed in the form†

$$(7) \qquad \begin{gathered} \frac{n}{D_1} \begin{vmatrix} y_a(x) & y_b(x) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} - \frac{m}{D_1} \begin{vmatrix} y_a(x) & y_b(x) \\ y_a(x_1) & y_b(x_1) \end{vmatrix}, \\ D_1 = \begin{vmatrix} y_a(x_1) & y_b(x_1) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} \neq 0.\ddagger \end{gathered}$$

We shall say that E satisfies the condition IV'_{Lb} if constants $\delta > 0$, $\eta > 0$, and A exist such that§

$$\Delta(x, x_1, y, \alpha) \equiv \begin{vmatrix} \bar{y}_a(x) & \bar{y}_b(x) \\ \bar{y}_a(x_1) & \bar{y}_b(x_1) \end{vmatrix}$$

is, in absolute value, greater than δ in the region $x_1 < x \leqq x_2$, $|y - y_1| \leqq \eta$, $A \geqq \alpha > -a^2$. The first determinant in (7) has a finite limit as n approaches zero; and hence, if n is small in absolute value, IV'_{Lb} insures that the expression will not vanish and that no two extremals of the family pass through the same

* a, b, $\bar{a}$, and $\bar{b}$ also depend on m and n, which are omitted in the notation.

† $y_a(x), \cdots$ are written for $y_a[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha], \cdots$.

‡ The method used by Bolza, loc. cit., pp. 73–75, shows that $D_1 \neq 0$.

§ $\bar{y}_a(x), \cdots$ are written for $y_a[x, \bar{a}(y, \alpha), \bar{b}(y, \alpha)\alpha], \cdots$.

point. This condition also requires ϕ to be strictly monotone in λ for a given x and α, so that an extremal of the family passes through each point of a certain region F about E. The region F is a field and is independent of k (that is, of α).*

Finally, if E satisfies I, II_b, III_b, and IV'_{Lb}, we have $L(E) < L(C)$ for every $C \neq E$ in F. But

$$L(C) = J(C) + \epsilon, \qquad \epsilon > 0, \qquad \lim_{k \doteq 0} \epsilon = 0.$$

Furthermore $L(E) = J(E)$, so that $J(E) < J(C) + \epsilon$, and finally $J(E) \leqq J(C)$.

5. *Applications.* The line $y=0$ is an extremal for a problem involving any one of the following integrands:

$$f \equiv M(x, y) + N(x, y)y', \qquad M_y = N_x,$$

$$f \equiv x^2 + y^2 + yy',\dagger$$

$$f \equiv y'^4.$$

Our sufficient conditions for an improper minimum are met by $y=0$ in each case, but III′ is not met for any of them.

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* Condition IV'_{Lb} could be replaced by the following. There exist constants $\eta > 0$, $\xi > 0$, and A such that E satisfies III for $x_0 \leqq x \leqq x_2$, $x_0 = x_1 - \xi$ and such that $\Delta(x, x_0, y, \alpha) \neq 0$ for $x_0 = x_1 - \xi$, $x_0 < x \leqq x_2$, $|y - y_1| \leqq \eta$, $A \geqq \alpha > -a^2$. See Bolza, loc. cit., bottom p. 103.

† An example given by Bolza, loc. cit., p. 35.