A NOTE ON YOUNG-STIELTJES INTEGRALS*

BY F. G. DRESSEL

THEOREM 1. If \( f(x) \) is bounded and measurable Borel, and \( g_1(x), g_2(x) \) are of bounded variation, then the following equality holds:

\[
\int_0^1 f(x) d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x + 0)dg_2(x) + \int_0^1 f(x)g_2(x - 0)dg_1(x).
\]

Proof. In a recent article Evans† showed that if \( g_1(x) \) and \( g_2(x) \) have no common points of discontinuity, then

\[
\int_0^1 f(x) d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x).
\]

Therefore (1) holds if either \( g_1(x) \) or \( g_2(x) \) are continuous. It remains to show that the theorem holds when \( g_1(x) \) and \( g_2(x) \) are both step functions. Under these circumstances we have

\[
\int_0^1 f(x)g_1(x + 0)dg_2(x) + \int_0^1 f(x)g_2(x - 0)dg_1(x)
= \sum f(\alpha_i)g_1(\alpha_i + 0)\left[g_2(\alpha_i + 0) - g_2(\alpha_i - 0)\right]
+ \sum f(\alpha_i)g_2(\alpha_i - 0)\left[g_1(\alpha_i + 0) - g_1(\alpha_i - 0)\right]
= \sum f(\alpha_i)[g_1(\alpha_i + 0)g_2(\alpha_i + 0) - g_1(\alpha_i - 0)g_2(\alpha_i - 0)]
= \int_0^1 f(x) d[g_1(x)g_2(x)],
\]

where the summations are taken over all the discontinuities of \( g_1(x) \) and \( g_2(x) \).

The following lemmas are immediate applications of equation (1).

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**Lemma 1.** If \( f(x) \) is positive, bounded, and Borel measurable, and \( g_1(x), g_2(x) \) are monotone increasing, bounded, continuous on the left, then

\[
\int_0^1 f(x) d\left[g_1(x)g_2(x)\right] \leq \int_0^1 f(x) g_1(x) dg_2(x) + \int_0^1 f(x) g_2(x) dg_1(x).
\]

**Lemma 2.** If in Lemma 1 the \( g_1(x) \) and \( g_2(x) \) are monotone decreasing functions continuous on the right, the inequality sign in (2) is reversed.

W. C. Randels* used Lemma 1 in proving the existence of a solution of

\[
f(x) = m(x) + \lambda \int_0^x f(y) dK(x, y).
\]

In essentially the same manner, by making use of Lemma 2, we may prove the following theorem.

**Theorem 2.** If \( g(x) \) is of bounded variation and if (a) \( K(x, y) \) is Borel measurable in \( y \) for every \( x \), (b) \( K(x + 0, y) = K(x, y) \), (c) \( K(x, x) = 0 \), (d) \( |K(x_1, y) - K(x_2, y)| \leq |T(x_1) - T(x_2)| \), where the function \( T(x) \) is bounded and non-decreasing with \( x \), then

\[
f(x) = g(x) + \lambda \int_0^x K(x, y) df(y)
\]

has a unique solution of bounded variation.

Duke University

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