BRANCH-POINT MANIFOLDS ASSOCIATED WITH A LINEAR SYSTEM OF PRIMALS*

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1. *Introduction.* Linear ∞^α systems of primals in S_r have been treated† only for $\alpha=1, 2$. The properties of a linear system are obtained from the characteristics of the jacobian and of the branch-point manifold associated with the system. There are, at present, no means for deriving most of the characteristics of a singular primal or manifold in S_r, especially for $r>4$.

In this paper, a theorem is developed giving a set of characteristics of the branch-point manifolds of the system and its sub-systems. This is a step, not only toward the characterization of a general linear system in S_r, but also toward the study of singular manifolds which contain both nodal and cuspidal manifolds.‡

2. *Definitions and Basic Considerations.* In S_r, the linear ∞^r system, F_r, of primals is defined by the equation

$$\sum \lambda_i f_i = 0, \qquad (i = 1, 2, \cdots, r+1), \tag{1}$$

in which the f_i are general algebraic functions of order n in the $r+1$ homogeneous variables x_i. Then $f_i=0$ is the equation of a primal of order n without singularities in S_r.

The primals of F_r in the r-space (x) are in (1, 1) correspondence with the primes $\sum a_i y_i=0$, $(i=1, 2, \cdots, r+1)$, of an r-space (y). This correspondence is defined by the equations

$$\rho y_i = f_i, \qquad (i = 1, 2, \cdots, r+1).$$

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† T. R. Hollcroft, *Pencils of hypersurfaces*, American Journal of Mathematics, vol. 53 (1931), pp. 929–936; *Nets of manifolds in i dimensions*, Annali di Matematica, (4), vol. 5 (1927–28), pp. 261–267.

‡ These terms will be used: *node*, a double point of a manifold at which the quadric hypercone is entirely general; *nodal manifold of a manifold f*, a manifold for every point of which (except points on pinch and singular loci) the two tangent linear manifolds to f are distinct; *cuspidal manifold of f*, a manifold for all points of which the two tangent linear manifolds to f coincide; *cone* to mean *hypercone* for $r>3$.

To a point P of (y), considered as bearing ∞^{r-1} primes, corresponds n^r points of (x). These n^r points are the basis points of the ∞^{r-1} linear system of primals F_{r-1} in which the primals are in (1, 1) correspondence with the primes in (y) through P. Since (y) contains ∞^r points, F_r contains ∞^r linear systems F_{r-1}.

In the general case, to an S_k of (y), $(k \leqq r-1)$, considered as bearing ∞^{r-k-1} primes, corresponds in (x) the basis manifold M_k (of dimension k and order n^{r-k}) of an ∞^{r-k-1} linear system of primals F_{r-k-1} in which the primals are in (1, 1) correspondence with the primes of (y) through S_k. Since (y) contains $\infty^{(k+1)(r-k)}$ linear manifolds S_k, the system F_r contains $\infty^{(k+1)(r-k)}$ linear systems F_{r-k-1}.

The jacobian J of the linear system F_r is a primal of order $(r+1)(n-1)$. It is the locus of double points and contacts of primals of F_r. The jacobian J also contains the jacobian manifolds of all the linear systems of primals contained in F_r such that the jacobians of the systems F_{r-k-1} form a $(k+1)(r-k)$-parameter linear system of manifolds on J. Likewise J contains the singularities of higher order and contacts of higher order of primals of F_r and of all linear systems of primals contained in F_r. The jacobian J has no actual singularities, only apparent singular manifolds.

The (1, 1) correspondence between the primals of F_r and the primes of (y) establishes a $(1, n^r)$ involution between the points of (y) and (x), and J is the locus of coincidences of this involution. The image of J in (y) is the branch-point primal L, the locus of points such that all primals of each associated F_{r-1} have contact with a line at a point on J. The ∞^{r-1} contacts generate J.

L is also the envelope of primes of (y) which correspond to primals of F_r that have a node. To the points of contact of primes with L correspond uniquely the nodes, which lie on J.

The order μ_0 of L is the number of points in which J and $r-1$ primals of F_r intersect, that is, $\mu_0=(r+1)(n-1)n^{r-1}$.

The classes of L are defined as follows:

μ_1, the order of the tangent cone to L from a point;
μ_2, the order of the tangent cone to L from a line;

. . . .

μ_{k+1}, the order of the tangent cone to L from an S_k;

. . . .

μ_{r-1}, the number of tangent primes to L from an S_{r-2}.

3. *A Theorem Defining Branch-Point Manifolds of the F_{r-k-1}.* The primals of an F_{r-k-1} of F_r of (x) are in (1, 1) correspondence with the primes of S_{r-k-1}, a sub-space of (y). This establishes a (1, 1) correspondence between the points of S_{r-k-1} and the basis manifolds M_{k-1} of the $(r-k-2)$-parameter linear systems of primals contained in F_{r-k-1}. The locus of points of S_{r-k-1} for which all of the primals of the associated $(r-k-2)$-parameter linear systems have contact at one point with a line is the branch-point manifold L_{r-k-2} (primal of S_{r-k-1}) and the locus of contacts in (x) is the jacobian manifold J_{r-k-2}.

As shown in §2, the primals of an $(r-k-1)$-parameter linear system of primals belonging to F_r in (x) are in (1, 1) correspondence with the primes of (y) through an S_k. The $(k+1)$st class of L, μ_{k+1}, is the order of the tangent cone enveloped by primes through S_k tangent to L. To each such tangent prime corresponds a primal of F_{r-k-1} and of F_r with a node.

Consider any given S_{r-k-1} of (y). S_{r-k-1} intersects each of the primes through S_k in an S_{r-k-2}, which is a prime of S_{r-k-1}. The primals of F_{r-k-1} are in (1, 1) correspondence with these primes [S_{r-k-2} of (y)] of S_{r-k-1}.

Since the order of the tangent cone to L from S_k is μ_{k+1}, the section of this tangent cone by S_{r-k-1} is a manifold V_{r-k-2} of dimension $r-k-2$ and order μ_{k+1}. This manifold V_{r-k-2} is the envelope of the primes of S_{r-k-1} which are sections by S_{r-k-1} of the primes of (y) through S_k tangent to L. Therefore the primes in S_{r-k-1} enveloping V_{r-k-2} are in (1, 1) correspondence with the primals of F_{r-k-1} which have a node. But, as previously shown, the (1, 1) correspondence between the primals of F_{r-k-1} and the primes of S_{r-k-1} establish an involution in which the branch-point manifold L_{r-k-2} of S_{r-k-1} is defined as the envelope of primes of S_{r-k-1} which correspond uniquely to primals of F_{r-k-1} that have a node. Therefore, in S_{r-k-1},

$$L_{r-k-2} \equiv V_{r-k-2}.$$

This identity establishes the following theorem.*

The section by an S_{r-k-1} of the tangent cone from an S_k to L, where L is the branch-point primal in the r-space (y) associated

* This theorem has been established for three dimensions. See T. R. Hollcroft, *The general web of algebraic surfaces of order n and the involution defined by it*, Transactions of this Society, vol. 35 (1933), p. 859.

with an r-parameter linear system of primals F_r of an r-space (x), is the branch-point manifold L_{r-k-2} of S_{r-k-1} associated with a linear $(r-k-1)$-parameter system of primals F_{r-k-1} belonging to F_r.

The order μ_{k+1} of L_{r-k-2} is also the order of the contour manifolds on L of the tangent cones from an S_k. These contour manifolds, of dimension $r-k-2$, form a linear system on L and are the respective images of the jacobian manifolds of the F_{r-k-1} contained in F_r. These jacobian manifolds form a linear system on J of the same respective dimension as the associated linear system of contour manifolds on L. Its contour manifold, L_{r-k-2}, and its associated jacobian manifold are all in (1,1) correspondence.

4. *Relations Resulting from the Theorem.* By the above theorem, the $(k+1)$st class μ_{k+1} of L is the order of the branch-point manifold L_{r-k-2} associated with an F_{r-k-1} belonging to F_r.

In the $(1, n^{r-k-1})$ involution associated with F_{r-k-1}, the condition for a point to lie on L_{r-k-2} is that the primals of F_{r-k-1} have a common tangent S_{k+2} at a common point. The condition that $r-k-1$ primals have a common tangent S_{k+2} at a common point is the tact-invariant of this system of primals. The order of this tact-invariant is*

$$\mu_{k+1}=\frac{1}{(k+2)!}(r+1)r(r-1)(r-2)\cdots(r-k)(n-1)^{k+2}n^{r-k-2}.$$

This is, therefore, the order of L_{r-k-2} and the value of μ_{k+1}, the $(k+1)$st class of L.

The order μ_0 of L results from the above formula for $k=-1$, that is, the order μ_0 is the tact-invariant of r primals of F_r. The final class of L, $\mu_{r-1}=(r+1)(n-1)^r$, is the order of the discriminant of a primal of F_r and is not a tact-invariant, since it involves only one primal. The value of μ_{r-1}, however, is also given by the above formula for $k=r-2$.

The class μ_{r-2} of L is the order of the tangent cone to L from an S_{r-3}. This is also the order of the branch-point curve L_1 associated with a net of primals of F_r. The complete set of charac-

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* T. R. Hollcroft, *Tact-invariants of primals in S_r*, Journal of the London Mathematical Society, vol. 11 (1936), p. 24.

teristics of L_1 is given in a former paper.* These are also the characteristics of a tangent cone (surface) to L from an S_{r-3}. The characteristics of L_2 and therefore of the tangent cone to L from an S_{r-4} have been found† for $n=2$, but not for a general n.

Since the final class‡ of a section of L made by an S_{k+2} is μ_{k+1}, the above value of μ_{k+1} gives the final classes of all sections of L by a linear manifold as well as the orders of all tangent cones to L from a linear manifold. The order of the section of L by any linear manifold is μ_0.

In general, L in (y) has both a nodal and a cuspidal manifold, each of dimension $r-2$, and these manifolds are themselves singular. For a linear system of dimension r in S_{r-1}, however, L has only a nodal manifold of dimension $r-2$, containing a pinch manifold of dimension $r-3$.

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* T. R. Hollcroft, *Nets of manifolds in i dimensions*, loc. cit.

† T. R. Hollcroft, *The web of quadric hypersurfaces in r dimensions*, this Bulletin, vol. 41 (1935), pp. 97–103.

‡ By *final class* of an S_{k+2} section of L is meant the number of S_{k-1} through an arbitrary S_k (all in S_{k+2}) tangent to L.