BRANCH-POINT MANIFOLDS ASSOCIATED WITH
A LINEAR SYSTEM OF PRIMALS*

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1. Introduction. Linear \( \alpha \) systems of primais in \( S_r \) have been treated only for \( \alpha = 1, 2 \). The properties of a linear system are obtained from the characteristics of the jacobian and of the branch-point manifold associated with the system. There are, at present, no means for deriving most of the characteristics of a singular primal or manifold in \( S_r \), especially for \( r > 4 \).

In this paper, a theorem is developed giving a set of characteristics of the branch-point manifolds of the system and its sub-systems. This is a step, not only toward the characterization of a general linear system in \( S_r \), but also toward the study of singular manifolds which contain both nodal and cuspidal manifolds.‡

2. Definitions and Basic Considerations. In \( S_r \), the linear \( \infty^r \) system, \( F_r \), of primais is defined by the equation

\[
\sum \lambda_i f_i = 0, \quad (i = 1, 2, \ldots, r+1),
\]

in which the \( f_i \) are general algebraic functions of order \( n \) in the \( r+1 \) homogeneous variables \( x_i \). Then \( f_i = 0 \) is the equation of a primal of order \( n \) without singularities in \( S_r \).

The primais of \( F_r \) in the \( r \)-space \((x)\) are in \((1, 1)\) correspondence with the primes \( \sum a_i y_i = 0, \quad (i = 1, 2, \ldots, r+1) \), of an \( r \)-space \((y)\). This correspondence is defined by the equations

\[
\rho y_i = f_i, \quad (i = 1, 2, \ldots, r+1).
\]

* Presented to the Society, September 12, 1935.
‡ These terms will be used: *node*, a double point of a manifold at which the quadric hypercone is entirely general; *nodal manifold of a manifold f*, a manifold for every point of which (except points on pinch and singular loci) the two tangent linear manifolds to \( f \) are distinct; *cuspidal manifold of f*, a manifold for all points of which the two tangent linear manifolds to \( f \) coincide; *cone* to mean *hypercone* for \( r > 3 \).
To a point $P$ of $(y)$, considered as bearing $\infty^{r-1}$ primes, corresponds $n^r$ points of $(x)$. These $n^r$ points are the basis points of the $\infty^{r-1}$ linear system of primals $F_{r-1}$ in which the primals are in $(1, 1)$ correspondence with the primes in $(y)$ through $P$. Since $(y)$ contains $\infty^r$ points, $F_r$ contains $\infty^r$ linear systems $F_{r-1}$.

In the general case, to an $S_k$ of $(y)$, $(k \leq r-1)$, considered as bearing $\infty^{r-k-1}$ primes, corresponds in $(x)$ the basis manifold $M_k$ (of dimension $k$ and order $n^{r-k}$) of an $\infty^{r-k-1}$ linear system of primals $F_{r-k-1}$ in which the primals are in $(1, 1)$ correspondence with the primes of $(y)$ through $S_k$. Since $(y)$ contains $\infty^{(k+1)(r_k)}$ linear manifolds $S_k$, the system $F_r$ contains $\infty^{(k+1)(r-k)}$ linear systems $F_{r-k-1}$.

The jacobian $J$ of the linear system $F_r$ is a primal of order $(r+1)(n-1)$. It is the locus of double points and contacts of primals of $F_r$. The jacobian $J$ also contains the jacobian manifolds of all the linear systems of primals contained in $F_r$ such that the Jacobians of the systems $F_{r-k-1}$ form a $(k+1)(r-k)$-parameter linear system of manifolds on $J$. Likewise $J$ contains the singularities of higher order and contacts of higher order of primals of $F_r$ and of all linear systems of primals contained in $F_r$. The jacobian $J$ has no actual singularities, only apparent singular manifolds.

The $(1, 1)$ correspondence between the primals of $F_r$ and the primes of $(y)$ establishes a $(1, n^r)$ involution between the points of $(y)$ and $(x)$, and $J$ is the locus of coincidences of this involution. The image of $J$ in $(y)$ is the branch-point primal $L$, the locus of points such that all primals of each associated $F_{r-1}$ have contact with a line at a point on $J$. The $\infty^{r-1}$ contacts generate $J$.

$L$ is also the envelope of primes of $(y)$ which correspond to primals of $F_r$ that have a node. To the points of contact of primes with $L$ correspond uniquely the nodes, which lie on $J$.

The order $\mu_0$ of $L$ is the number of points in which $J$ and $r-1$ primals of $F_r$ intersect, that is, $\mu_0 = (r+1)(n-1)n^{r-1}$.

The classes of $L$ are defined as follows:

- $\mu_1$, the order of the tangent cone to $L$ from a point;
- $\mu_2$, the order of the tangent cone to $L$ from a line;
- $\mu_{k+1}$, the order of the tangent cone to $L$ from an $S_k$;
- $\mu_{r-1}$, the number of tangent primes to $L$ from an $S_{r-2}$. 

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3. A Theorem Defining Branch-Point Manifolds of the $F_{r-k-1}$.

The primals of an $F_{r-k-1}$ of $F_r$ of $(x)$ are in $(1, 1)$ correspondence with the primes of $S_{r-k-1}$, a sub-space of $(y)$. This establishes a $(1, 1)$ correspondence between the points of $S_{r-k-1}$ and the basis manifolds $M_{k-1}$ of the $(r-k-2)$-parameter linear systems of primals contained in $F_{r-k-1}$. The locus of points of $S_{r-k-1}$ for which all of the primals of the associated $(r-k-2)$-parameter linear systems have contact at one point with a line is the branch-point manifold $L_{r-k-2}$ (primal of $S_{r-k-1}$) and the locus of contacts in $(x)$ is the jacobian manifold $J_{r-k-2}$.

As shown in §2, the primals of an $(r-k-1)$-parameter linear system of primals belonging to $F_r$ in $(x)$ are in $(1, 1)$ correspondence with the primes of $(y)$ through an $S_k$. The $(k+1)$st class of $L$, $\mu_{k+1}$, is the order of the tangent cone enveloped by primes through $S_k$ tangent to $L$. To each such tangent prime corresponds a primal of $F_{r-1}$ and of $F_r$ with a node.

Consider any given $S_{r-k-1}$ of $(y)$. $S_{r-k-1}$ intersects each of the primes through $S_k$ in an $S_{r-k-2}$, which is a prime of $S_{r-k-1}$. The primals of $F_{r-k-1}$ are in $(1, 1)$ correspondence with these primes $[S_{r-k-2}$ of $(y)]$ of $S_{r-k-1}$.

Since the order of the tangent cone to $L$ from $S_k$ is $\mu_{k+1}$, the section of this tangent cone by $S_{r-k-1}$ is a manifold $V_{r-k-2}$ of dimension $r-k-2$ and order $\mu_{k+1}$. This manifold $V_{r-k-2}$ is the envelope of the primes of $S_{r-k-1}$ which are sections by $S_{r-k-1}$ of the primes of $(y)$ through $S_k$ tangent to $L$. Therefore the primes in $S_{r-k-1}$ enveloping $V_{r-k-2}$ are in $(1, 1)$ correspondence with the primals of $F_{r-k-1}$ which have a node. But, as previously shown, the $(1, 1)$ correspondence between the primals of $F_{r-k-1}$ and the primes of $S_{r-k-1}$ establish an involution in which the branch-point manifold $L_{r-k-2}$ of $S_{r-k-1}$ is defined as the envelope of primes of $S_{r-k-1}$ which correspond uniquely to primals of $F_{r-k-1}$ that have a node. Therefore, in $S_{r-k-1}$,

$$L_{r-k-2} = V_{r-k-2}.$$  

This identity establishes the following theorem.*

*This theorem has been established for three dimensions. See T. R. Hollcroft, *The general web of algebraic surfaces of order n and the involution defined by it*, Transactions of this Society, vol. 35 (1933), p. 859.
with an \( r \)-parameter linear system of primais \( F_r \) of an \( r \)-space \( (x) \), is the branch-point manifold \( L_{r-k-2} \) of \( S_{r-k-1} \) associated with a linear \( (r-k-1) \)-parameter system of primais \( F_{r-k-1} \) belonging to \( F_r \).

The order \( \mu_{k+1} \) of \( L_{r-k-2} \) is also the order of the contour manifolds on \( L \) of the tangent cones from an \( S_k \). These contour manifolds, of dimension \( r-k-2 \), form a linear system on \( L \) and are the respective images of the jacobian manifolds of the \( F_{r-k-1} \) contained in \( F_r \). These jacobian manifolds form a linear system on \( J \) of the same respective dimension as the associated linear system of contour manifolds on \( L \). Its contour manifold, \( L_{r-k-2} \), and its associated jacobian manifold are all in \((1,1)\) correspondence.

4. Relations Resulting from the Theorem. By the above theorem, the \((k+1)\)st class \( \mu_{k+1} \) of \( L \) is the order of the branch-point manifold \( L_{r-k-2} \) associated with an \( F_{r-k-1} \) belonging to \( F_r \).

In the \((1, n^{r-k-1})\) involution associated with \( F_{r-k-1} \), the condition for a point to lie on \( L_{r-k-2} \) is that the primais of \( F_{r-k-1} \) have a common tangent \( S_{k+2} \) at a common point. The condition that \( r-k-1 \) primais have a common tangent \( S_{k+2} \) at a common point is the tact-invariant of this system of primais. The order of this tact-invariant is

\[
\mu_{k+1} = \frac{1}{(k+2)!} (r+1)r(r-1)(r-2) \cdots (r-k)(n-1)^{k+2}M^{r-k-2}.
\]

This is, therefore, the order of \( L_{r-k-2} \) and the value of \( \mu_{k+1} \), the \((k+1)\)st class of \( L \).

The order \( \mu_0 \) of \( L \) results from the above formula for \( k = -1 \), that is, the order \( \mu_0 \) is the tact-invariant of \( r \) primais of \( F_r \). The final class of \( L \), \( \mu_{r-1} = (r+1)(n-1)^r \), is the order of the discriminant of a primal of \( F_r \) and is not a tact-invariant, since it involves only one primal. The value of \( \mu_{r-1} \), however, is also given by the above formula for \( k = r-2 \).

The class \( \mu_{r-2} \) of \( L \) is the order of the tangent cone to \( L \) from an \( S_{r-3} \). This is also the order of the branch-point curve \( L_1 \) associated with a net of primais of \( F_r \). The complete set of charac-

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teristics of $L_1$ is given in a former paper.* These are also the characteristics of a tangent cone (surface) to $L$ from an $S_{r-3}$. The characteristics of $L_2$ and therefore of the tangent cone to $L$ from an $S_{r-4}$ have been found† for $n = 2$, but not for a general $n$.

Since the final class‡ of a section of $L$ made by an $S_{k+2}$ is $\mu_{k+1}$, the above value of $\mu_{k+1}$ gives the final classes of all sections of $L$ by a linear manifold as well as the orders of all tangent cones to $L$ from a linear manifold. The order of the section of $L$ by any linear manifold is $\mu_0$.

In general, $L$ in $(y)$ has both a nodal and a cuspidal manifold, each of dimension $r-2$, and these manifolds are themselves singular. For a linear system of dimension $r$ in $S_{r-1}$, however, $L$ has only a nodal manifold of dimension $r-2$, containing a pinch manifold of dimension $r-3$.

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‡ By final class of an $S_{k+2}$ section of $L$ is meant the number of $S_{k-1}$ through an arbitrary $S_k$ (all in $S_{k+2}$) tangent to $L$. 