

A NOTE ON THE RELATION BETWEEN INTEGRAL  
AND TCHEBYCHEFF APPROXIMATION BY  
POLYNOMIALS IN THE COMPLEX  
DOMAIN\*

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1. *Introduction.* Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $R$  be the limited simply connected region bounded by  $C$ . Let  $f(z)$  be analytic in  $R$  and continuous in  $R + C \equiv \bar{C}$ , and let  $P_n(z)$  be a polynomial of degree  $n$  in  $z$  which minimizes the integral

$$(1) \quad \int_C w(z) |f(z) - P_n(z)|^p |dz|,$$

where  $p$  is a fixed positive number, and  $w(z)$  is a bounded non-negative measurable function bounded from zero; the existence of such a polynomial  $P_n(z)$  is well known.† Dunham Jackson‡ has given an evaluation for  $|f(z) - P_n(z)|$ ,  $z$  in  $\bar{C}$ , with various restrictions on  $C$  and on  $R$ . In this note we sharpen these results for curves with corners, and extend them to more general smooth curves and to arbitrary rectifiable Jordan curves.

We also consider the case  $p = 2$  in particular and the development of  $f(z)$  of class  $L^2$  in normal and orthogonal polynomials. Our two principal results are the following theorems.

**THEOREM A.** *Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $f(z)$  be analytic in  $C$  and continuous in  $\bar{C}$ . Let  $p_n(z)$  be an arbitrary polynomial of degree  $n$  such that  $|f(z) - p_n(z)| \leq \epsilon_n$ ,  $z$  in  $\bar{C}$ . Then  $|f(z) - P_n(z)| \leq Mn^{2/p}\epsilon_n$ ,  $p > 0$ ,  $z$  in  $\bar{C}$ , where  $M$  is a constant independent of  $n$  and  $z$ , and  $P_n(z)$  is a polynomial of degree  $n$  which minimizes (1).*

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† See, for example, J. L. Walsh, *Interpolation and Approximation*, Colloquium Publications of this Society, vol. 20, 1935, pp. 351-352.

‡ *On certain problems of approximation in the complex domain*, this Bulletin, vol. 36 (1930), pp. 851-857; *On the application of Markoff's theorem to problems of approximation in the complex domain*, this Bulletin, vol. 37 (1931), pp. 883-890. These papers will be referred to hereafter as JI and JII, respectively.

**THEOREM B.** *Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $f(z)$  belong to  $L^2$  on  $C$ . Let  $\{P_n(z)\}$  be the set of polynomials normal and orthogonal on  $C$  and let\**

$$a_k = \int_C f(z) \overline{P_k(z)} |dz|.$$

*If  $\sum_{k=0}^{\infty} |a_k| k$  converges, the function  $f_1(z) \equiv \sum_{k=0}^{\infty} a_k P_k(z)$  is analytic in  $C$ , continuous in  $\overline{C}$ , and  $f_1(z) = f(z)$  almost everywhere on  $C$ .*

The method is an application of recent results of the author† on the modulus of the derivative of a polynomial and is the same as that used by Jackson in *JI* and *JII*.

It should be noted here that in §3 (Theorem A) the region may be a multiply connected region bounded by a finite number of rectifiable Jordan curves, or made up of a finite number of separate regions of similar character.‡

2. *Jordan Curves and Derivatives of Polynomials.* § Let  $R$ , with boundary  $C$ , be a limited simply connected region in the  $z$ -plane and let  $z = \psi(w)$  map  $K$ , the complement (with respect to the extended plane) of  $\overline{C}$ , on  $|w| > 1$  so that the points at  $\infty$  in the two planes correspond to each other. We will say that  $C$  is a curve of Type *S*|| if

$$(2) \quad 0 < N_1 < \left| \frac{\psi(w_1) - \psi(w_2)}{w_1 - w_2} \right| < N_2 < \infty, \\ (|w_1| \geq 1, |w_2| \geq 1),$$

where  $N_1$  and  $N_2$  are constants independent of  $w_1$  and  $w_2$ . If  $C$  is a curve of Type *S* it is shown in *SII* that for  $P_n(z)$ , an arbitrary polynomial of degree  $n$ , the inequality  $|P_n(z)| \leq M$ ,  $z$  on  $C$ , im-

\*  $\bar{a}$  denotes the conjugate of the complex number  $a$ .

† On the modulus of the derivative of a polynomial, this Bulletin, vol. 42 (1936), pp. 699–702; *Generalized derivatives and approximation by polynomials*, Transactions of this Society, vol. 41 (1937), pp. 84–123. These papers will be referred to hereafter as *SI* and *SII*, respectively.

‡ See *JII*, p. 885.

§ The results given here are extensions of Bernstein's and Markoff's Theorems on the moduli of the derivatives of polynomials; see *SII* for references.

|| For the geometric properties of  $C$  see W. Seidel, *Über die Ränderzuordnung bei konformen Abbildungen*, *Mathematische Annalen*, vol. 104 (1931), pp. 182–243; especially pp. 217–221.

plies\*  $|P'_n(z)| \leq MM_1n$ ,  $z$  on  $C$ ; here, and below  $M$  and  $M_1$  are constants independent of  $n$  and  $z$ ; the constant  $M_1$  depends on  $C$ .

For curves with corners we need the following definition (see SII).

**DEFINITION.** Let  $C$  be a Jordan curve composed of a finite number of Jordan arcs meeting in corners  $z_1, z_2, \dots, z_r$ , of exterior openings  $\mu_1\pi, \mu_2\pi, \dots, \mu_r\pi$ , ( $2 > \mu_1 \geq \mu_2 \geq \dots \geq \mu_r > 0$ ), and let the difference quotient of the mapping function [see (2)] be bounded in modulus on each sub-arc not containing a corner. Let  $t = \mu_1$  if  $\mu_1 \geq 1$ , and  $t = 1$  if  $\mu_1 < 1$ . Then we shall say that  $C$  is a curve of Type  $t$ .

If  $C$  is a curve of Type  $t$ , the inequality  $|P_n(z)| \leq M$ ,  $z$  on  $C$ , implies  $|P'_n(z)| \leq MM_1n^t$ ,  $z$  on  $C$  (see SII).

If  $C$  is a rectifiable Jordan curve it is shown in SI that  $|P_n(z)| \leq M$ ,  $z$  on  $C$ , implies  $|P'_n(z)| \leq MM_1n^2$ ,  $z$  on  $C$ .

3. *Integral and Tchebycheff Approximation.* For approximation in the sense of least  $p$ th powers we have the following theorem.

**THEOREM 1.** Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $f(z)$  be analytic in  $C$  and continuous in  $\bar{C}$ . Let  $P_n(z)$ , ( $n = 1, 2, \dots$ ), be a polynomial of degree  $n$  which minimizes

$$\int_C w(z) |f(z) - P_n(z)|^p |dz|,$$

where  $w(z)$  is a bounded positive measurable function, with a positive lower bound, and  $p$  is a fixed positive number. Let  $p_n(z)$  be a polynomial of degree  $n$  such that  $|f(z) - p_n(z)| \leq \epsilon_n$ ,  $z$  in  $\bar{C}$ . Then we have

$$(3) \quad |f(z) - P_n(z)| \leq Mn^{t/p}\epsilon_n, \quad (z \text{ in } \bar{C}),$$

where  $M$  is a constant independent of  $n$  and  $z$ , and  $t = 1$  if  $C$  is a curve of Type  $S$ ,  $1 \leq t < 2$  if  $C$  is a curve of Type  $t$ , and  $t = 2$  if  $C$  is an arbitrary rectifiable Jordan curve.

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\*  $f'(z)$  denotes the first derivative of  $f(z)$ .

We indicate the proof (see JI). Let  $r_n(z) = f(z) - p_n(z)$ , and let  $\pi_n(z) = P_n(z) - p_n(z)$ ; then  $r_n(z) - \pi_n(z) = f(z) - P_n(z)$ . Let

$$\begin{aligned} \gamma_n &= \int_C w(z) |f(z) - P_n(z)|^p |dz| \\ &= \int_C w(z) |r_n(z) - \pi_n(z)|^p |dz|. \end{aligned}$$

Let  $|\pi_n(z)| \leq \mu_n$ ,  $z$  on  $C$ , and let  $|\pi_n(z_0)| = \mu_n$ , where  $z_0$  is a point of  $C$ . Then by the results of §2 we know that  $|\pi_n'(z)| \leq \mu_n M_1 n^t$ ,  $z$  on  $C$ , and it follows that

$$|\pi_n(z) - \pi_n(z_0)| \leq \mu_n M_1 n^t |z - z_0|,$$

$z$  on  $C$ . Let  $s$  be the arc or arcs of  $C$  consisting of the set of points  $\zeta$ :  $\zeta$  on  $C$ ,  $|\zeta - z_0| < 1/(2M_1 n^t)$ . On  $s$  we have  $|\pi_n(\zeta) - \pi_n(z_0)| \leq \mu_n/2$ . This means that on  $s$ , whose total length is not less than  $1/(M_1 n^t)$ , we have  $|\pi_n(\zeta)| \geq \mu_n/2$ . Let  $V \geq w(z) \geq v > 0$ , and we have for  $\mu_n \geq 4\epsilon_n$

$$(4) \quad \gamma_n \geq v \left(\frac{\mu_n}{4}\right)^p \frac{1}{M_1 n^t},$$

and by the minimizing property of  $P_n(z)$ , we know that

$$(5) \quad \gamma_n \leq LV\epsilon_n^p,$$

where  $L$  is the length of  $C$ . Combining inequalities (4) and (5) yields  $\mu_n \leq M_2 n^{t/p} \epsilon_n$ , where  $M_2$  is a constant depending on  $C$ ,  $p$ , and  $w(z)$ , but independent of  $n$  and  $z$ . Thus whether  $\mu_n \geq \epsilon_n$  or not we have

$$|f(z) - P_n(z)| = |r_n(z) - \pi_n(z)| \leq M n^{t/p} \epsilon_n, \quad (z \text{ in } \bar{C}),$$

where  $M$  is independent of  $n$  and  $z$ , and the proof is complete.

Jackson (JI) establishes Theorem 1 with  $t=1$  for  $C$  satisfying the condition that there is a number  $r_0 > 0$  such that at every point of  $C$  a circle of radius  $r_0$  can be drawn tangent to  $C$ , and containing in its interior and on its boundary only points of  $\bar{C}$ .\* He also obtains (JII) the result with  $t=2$  for the case where  $C$  is a curve such that its parametric representation satisfies a

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\* For a discussion of such properties see Seidel, loc. cit.

Lipschitz condition\* of order 1 and  $\mathcal{R}$  is a region for which there is a positive number  $r_0$  such that from every point of its boundary a line segment of length  $2r_0$  can be drawn belonging wholly to the closed region  $\overline{\mathcal{R}}$ .

If  $C$  is an analytic Jordan curve and  $f^{(j)}(z)$ , ( $j \geq 0$ ), satisfies † a Lipschitz condition of order  $\alpha$ , ( $0 < \alpha \leq 1$ ), on  $\overline{C}$ , we know by a theorem of John Curtiss‡ that  $p_n(z)$  exists such that  $\epsilon_n \leq M_1/n^{j+\alpha}$ , where  $M_1$  is a constant independent of  $n$  and  $z$ . Consequently, in this case we have by inequality (3)

$$(6) \quad |f(z) - P_n(z)| \leq \frac{M}{n^{j+\alpha-1/p}}, \quad (z \text{ in } \overline{C});$$

and hence if  $j+\alpha > 1/p$ , we have uniform convergence of the sequence  $P_n(z)$  to  $f(z)$  in  $\overline{C}$ . In fact (6) gives an upper bound on the degree of Tchebycheff approximation of  $P_n(z)$ .

In connection with (3) it is interesting to note that for each  $n$  we have §

$$\lim_{p \rightarrow \infty} P_n(z) = T_n(z),$$

where  $P_n(z)$  is the polynomial of degree  $n$  of best approximation to the continuous function  $f(z)$  on  $C$ , a rectifiable Jordan curve, in the sense of least  $p$ th powers with a norm function, || and  $T_n(z)$  is the polynomial of degree  $n$  of best approximation to  $f(z)$  on  $C$  in the sense ¶ of Tchebycheff.

\*  $f(z)$  satisfies a Lipschitz condition of order  $\alpha$  on the set  $E$  if for arbitrary points  $z_1$  and  $z_2$  on  $E$  we have  $|f(z_1) - f(z_2)| \leq L|z_1 - z_2|^\alpha$ , where  $L$  is a constant independent of  $z_1$  and  $z_2$ .

†  $f^{(0)}(z) \equiv f(z)$ .

‡ *A note on the degree of polynomial approximation*, this Bulletin, vol. 42 (1936), pp. 873-878.

§ G. Julia, *Sur les polynomes de Tchebycheff*, Comptes Rendus (Paris), vol. 182 (1926), pp. 1201-1202; the corresponding result for  $C$  the segment  $(0, 1)$  of the axis of reals is due to G. Pólya, *Sur un algorithme toujours convergent pour obtenir les polynomes de meilleure approximation de Tchebycheff pour une fonction continue quelconque*, *ibid.*, vol. 153 (1915), pp. 840-843.

¶ See J. L. Walsh, *Approximation by Polynomials in the Complex Domain*, Mémorial des Sciences Mathématiques, vol. 73 (1935); especially p. 24.

¶ That is,  $\max [|f(z) - T_n(z)|, z \text{ on } C]$  is less than the corresponding expression for any other polynomial of degree  $n$ .

4. *The Case\**  $p=2$ . Approximation in the sense of least squares leads to a consideration of the set of polynomials  $\{P_n(z)\}$  normal and orthogonal on  $C$ , a rectifiable Jordan curve in the  $z$ -plane. The method used in proving Theorem 1 serves to establish the following result.

**THEOREM 2.** *Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $\int_C |Q_n(z)|^p dz = \epsilon_n$ ,  $p > 0$ , where  $Q_n(z)$  is a polynomial of degree  $n$ . Then we have*

$$(7) \quad |Q_n(z)| \leq Mn^{t/p}\epsilon_n^{1/p}, \quad (z \text{ on } C),$$

where  $M$  is a constant independent of  $n$  and  $z$ , and  $t=1$  if  $C$  is a curve of Type  $S$ ,  $1 \leq t < 2$  if  $C$  is a curve of Type  $t$ , and  $t=2$  if  $C$  is an arbitrary rectifiable Jordan curve.

For the set of polynomials  $\{P_n(z)\}$  normal and orthogonal on  $C$ , we have  $\epsilon_n = 1$  with  $p=2$  in Theorem 2. Thus (7) becomes

$$|P_n(z)| \leq Mn^{t/2}, \quad (z \text{ on } C).$$

Now suppose  $f(z)$  belongs to  $L^2$  on  $C$ ; then

$$f(z) \sim a_0P_0(z) + a_1P_1(z) + \dots + a_kP_k(z) + \dots,$$

$$a_k = \int_C f(z)\overline{P_k(z)} dz,$$

where the sign  $\sim$  is used to denote formal correspondence. The polynomial of degree  $n$  of best approximation to  $f(z)$  on  $C$  in the sense of least squares is  $S_n(z) = \sum_{k=0}^n a_k P_k(z)$ . Now suppose  $\sum_{k=0}^{\infty} |a_k| k^{t/2}$ , where the value of  $t$  is compatible with the character of  $C$ , converges, then

$$f_1(z) = \sum_{k=0}^{\infty} a_k P_k(z)$$

converges absolutely and uniformly on, and hence within,  $C$  and consequently  $f_1(z)$  is analytic in  $C$  and continuous in  $\overline{C}$ . Moreover, since  $S_n(z)$  converges in the mean to  $f(z)$  on  $C$ , the function  $f_1(z) = f(z)$  almost everywhere on  $C$ . Thus we have the following theorem.

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\* See G. Szegő, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, *Mathematische Zeitschrift*, vol. 9 (1921), pp. 218–270; J. L. Walsh, *Interpolation and Approximation*, Chap. VI.

**THEOREM 3.** *Let  $C$  be a rectifiable Jordan curve in the  $z$ -plane and let  $f(z)$  belong to  $L^2$  on  $C$ . Let  $\{P_n(z)\}$  be the set of polynomials normal and orthogonal on  $C$  and let*

$$f_1(z) = a_0P_0(z) + a_1P_1(z) + \cdots + a_kP_k(z) + \cdots ,$$

$$a_k = \int_C f(z)\overline{P_k(z)} |dz| .$$

*Suppose  $\sum_{k=0}^{\infty} |a_k| k^{t/2}$ , ( $2 \geq t \geq 1$ ), converges . Then  $f_1(z)$  is analytic in  $C$ , continuous in  $\overline{C}$ , and is equal to  $f(z)$  almost everywhere on  $C$  if either: (1)  $C$  is a curve of Type  $S$  and  $t=1$ , or (2)  $C$  is curve of Type  $t$  and  $2 > t \geq 1$ , or (3)  $C$  is an arbitrary rectifiable Jordan curve and  $t=2$ .*

Of course in the above theorem the function  $f(z)$  may be defined (or redefined) on a set of measure 0 on  $C$  and defined in  $C$  so as to coincide everywhere with  $f_1(z)$ .

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