SOME INEQUALITIES CONCERNING FUNCTIONS OF EXPONENTIAL TYPE

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In this paper we derive two inequalities concerning entire functions satisfying the conditions $f(z)=O(e^{n|z|})$, and $|f(z)|\leqq 1$ on the real axis. Our results are closely related to theorems given by S. Bernstein, Szegö, van der Corput and Schaake, and Boas.*

THEOREM I. *Let $f(z)$ be an entire function such that*

$$f(z) = O(e^{n|z|}) \tag{1}$$

uniformly over the entire plane, and on the real axis

$$|f(z)| \leqq 1. \tag{2}$$

Then, if a and b are any two real numbers,

$$|af(z) + bf'(z)| \leqq (a^2 + n^2b^2)^{1/2} \tag{3}$$

for real z.

PROOF OF THEOREM I: By means of Cauchy's Integral Formula and the Phragmén-Lindelöf Principle it is not difficult to show that if $f(z)$ satisfies the conditions of Theorem I then

$$\frac{d}{dz}\left\{\frac{f(z)}{\sin n(z+\alpha)}\right\} = -n\sum_{k=-\infty}^{\infty}\frac{(-1)^k f\left(\frac{k\pi}{n}-\alpha\right)}{\{n(z+\alpha)-k\pi\}^2}. \tag{4}$$

For the proof of (4) see Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. II, page 35 and page 218. If α is real we have $|f(k\pi/n-\alpha)|\leqq 1$, hence for all real z and α

$$\begin{aligned} &|f'(z)\sin n(z+\alpha) - nf(z)\cos n(z+\alpha)| \\ &\qquad \leqq \sum_{-\infty}^{\infty}\frac{n\sin^2 n(z+\alpha)}{\{n(z+\alpha)-k\pi\}^2}. \end{aligned} \tag{5}$$

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* S. Bernstein, Comptes Rendus, vol. 176 (1923), p. 1603.

Van der Corput and Schaake, Compositio Matematica, vol. 2 (1935), p. 321; vol. 3 (1936), p. 128.

Szegö, Schriften der Königsberger gelehrten Gesellschaft, Naturwissenschaftliche Klasse, Fünftes Jahr 4 (1928), p. 69.

Boas, Transactions of this Society, vol. 40 (1936), p. 287.

Using the expansion $1/\sin^2 nw = \sum_{-\infty}^{\infty} 1/(nw - k\pi)^2$, we obtain

$$(6) \qquad | f'(z) \sin n(z + \alpha) - nf(z) \cos n(z + a) | \leqq n.$$

Given any real numbers a and b we may choose an α such that $\sin n(z+\alpha) = b/(a^2/n^2+b^2)^{1/2}$ and $-n \cos n(z+\alpha) = a/(a^2/n^2+b^2)^{1/2}$. Substitution in (6) then proves Theorem I.

THEOREM II. *Let $f(z)$ be a function which is real on the real axis and which satisfies the conditions of Theorem* I. *Then, if* $z = x+iy$,

$$(7) \qquad | f'(z) |^2 + n^2 | f(z) |^2 \leqq n^2 \cosh 2ny.$$

Unless $f(z)$ is of the form $\cos n(z+\alpha)$ *the equality sign can occur only at points on the real axis where $f = \pm 1$.*

PROOF OF THEOREM II: If $n(z+\alpha)$ is not a multiple of π the equality sign can occur in (5) only if $\pm f(k\pi/n - \alpha) = (-1)^k$ for all k. Putting these values for $f(k\pi/n - \alpha)$ in (4) one finds $f(z) \equiv \pm \cos n(z+\alpha)$. If $n(z+\alpha)$ is a multiple of π the equality sign in (5) clearly can occur only if $f(z) = \pm 1$.

In (3) let $a = n^2 f$ and $b = f'$. Then we find for real z

$$(8) \qquad \{f'(z)\}^2 + n^2\{f(z)\}^2 \leqq n^2.$$

We shall suppose throughout that $f(z)$ is not of the form $\cos n(z+\alpha)$ for any real α. Then the equality sign can occur in (8) only at points where $f(z) = \pm 1$. From (8) we have $|f'(z)| < n$ for real z. It is easy to show that $f'(z) = O(e^{n|z|})$, so that $f'(z)/n$ satisfies all the conditions that $f(z)$ does, and we have from (8)

$$(9) \qquad \left\{\frac{f''(z)}{n}\right\}^2 + n^2\left\{\frac{f'(z)}{n}\right\}^2 < n^2.$$

This is a strict inequality, for the equality can occur only at points where $f'(z)/n = \pm 1$, and we have shown that $|f'(z)| < n$.

In the same way one shows by induction that for the higher derivatives

$$\left\{\frac{f^{(k+1)}(z)}{n^k}\right\}^2 + n^2\left\{\frac{f^{(k)}(z)}{n^k}\right\}^2 < n^2.$$

Thus if $f(z)$ is expanded in a power series

$$f(z) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{\nu!} z^\nu,$$

we have $n^2a_\nu^2+a_{\nu+1}^2 \leqq n^{2(\nu+1)}$, or

$$|na_\nu + ia_{\nu+1}| \leqq n^{\nu+1}. \tag{10}$$

The equality sign can occur only if $\nu=0$.

Let us write $f(z)$ in the form

$$f(z) = g(z) + u(z) = \sum_0^\infty \frac{a_{2\nu}}{(2\nu)!} z^{2\nu} + \sum_0^\infty \frac{a_{2\nu+1}}{(2\nu+1)!} z^{2\nu+1}$$

where $g(z)$ is even and $u(z)$ is odd. We see that on the imaginary axis ($z=iy$) the functions $g(z)$ and $u'(z)$ are real while $g'(z)$ and $u(z)$ are pure imaginary. Let $|y|>0$. Then

$$n^2|f(iy)|^2 + |f'(iy)|^2 = n^2|g(iy)|^2 + n^2|u(iy)|^2 + |g'(iy)|^2 + |u'(iy)|^2.$$

Combining the two even functions g and u' we obtain

$$n^2|g(iy)|^2 + |u'(iy)|^2 = |ng(iy) + iu'(iy)|^2 = \left| \sum_0^\infty \frac{na_{2\nu} + ia_{2\nu+1}}{(2\nu)!} (iy)^{2\nu} \right|^2.$$

By the inequality (10) this is less than

$$\left| \sum_0^\infty \frac{n^{2\nu+1}}{(2\nu)!} y^{2\nu} \right|^2 = n^2 \cosh^2 ny.$$

Combining the odd functions u and g', we have

$$n^2|u(iy)|^2 + |g'(iy)|^2 = |nu(iy) + ig'(iy)|^2 = \left| \sum_0^\infty \frac{na_{2\nu+1} + ia_{2\nu+2}}{(2\nu+1)!} (iy)^{2\nu+1} \right|^2 < \left| \sum_0^\infty \frac{n^{2\nu+2}}{(2\nu+1)!} y^{2\nu+1} \right|^2 = n^2 \sinh^2 ny.$$

Thus we have

$$n^2|f(iy)|^2 + |f'(iy)|^2 < n^2\cosh^2 ny + n^2 \sinh^2 ny = n^2 \cosh 2ny.$$

We have thus demonstrated that (7) holds on the imaginary axis; but if β is real $f(z+\beta)$ also satisfies the conditions, so this is sufficient. We remark that if $f(z) = \cos n(z+\alpha)$, (7) becomes an equality throughout the plane.

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