

CONCERNING NORMAL AND COMPLETELY  
NORMAL SPACES\*

BY F. B. JONES

Urysohn has shown that any completely separable, normal topological space is metric. It is the principal object of this paper to establish a similar result for certain separable spaces.

**THEOREM 1.** *Every subset of power  $c$ † of a separable normal‡ Fréchet space-L (or -H) has a limit point.*

**PROOF.** Suppose, on the contrary, that  $S$  is a separable, normal Fréchet space-L (or -H) which contains a point set  $M$  of power  $c$  having no limit point. Let  $Z$  denote a countable subset of  $S$  such that every point of  $S$  either belongs to  $Z$  or is a limit point of  $Z$ . Since  $S$  is normal, there exists for each proper subset  $J$  of  $M$  a domain  $D_J$  which contains  $J$  but which neither contains a point of  $M - J$  nor has a limit point in  $M - J$ . If  $J$  and  $K$  are two different proper subsets of  $M$ , then  $Z \cdot D_J$  and  $Z \cdot D_K$  are different subsets of  $Z$ . Hence, there are at least as many subsets of  $Z$  as there are proper subsets of  $M$ . However, since  $M$  is of power  $c$  and  $Z$  is only countable, there are *more* than  $c$  proper subsets of  $M$  but at most  $c$  subsets of  $Z$ . This is a contradiction.

The above argument with slight changes establishes the following three theorems.

**THEOREM 2.** *Every subset of power  $c$  of a separable, completely normal§ Fréchet space-L (or -H) contains a limit point of itself.*

**THEOREM 3.** *If  $2^{\aleph_1} > 2^{\aleph_0}$ , every uncountable subset of a separable normal Fréchet space-L (or -H) has a limit point.||*

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† The number  $c$  is the power of the continuum.

‡ A space is said to be *normal* provided that, if  $P$  and  $Q$  are two mutually exclusive closed sets, there exist two mutually exclusive domains containing  $P$  and  $Q$  respectively.

§ A space is said to be *completely normal* provided that, if  $P$  and  $Q$  are two mutually separate point sets, there exist two mutually exclusive domains containing  $P$  and  $Q$  respectively.

|| The numbers  $\aleph_0$  and  $\aleph_1$  are the first and second transfinite cardinals respectively. That  $2^{\aleph_1} > 2^{\aleph_0}$  is an immediate consequence of a well known theorem if the *hypothesis of the continuum* holds true, that is, if  $\aleph_1 = c$ .

**THEOREM 4.** *If  $2^{\aleph_1} > 2^{\aleph_0}$ , every uncountable subset of a separable completely normal Fréchet space- $L$  (or  $-H$ ) contains a limit point of itself.*

A space- $L$  may, however, be separable and normal but contain an uncountable point set *not* containing a limit point of itself. This is shown by the following example. A lemma will be established first.

**LEMMA A.** *There exists on the number interval  $I(0, 1)$  a point set  $M$  of power  $\aleph_1$  such that if  $K$  is a countable subset of  $M$ , then  $K$  is an inner limiting set with respect to  $M$ .\**

**PROOF.** Let  $\alpha$  denote a well-ordered sequence whose elements are the points of  $I(0, 1)$ . Let  $P_1$  denote the first point of  $\alpha$ . Let  $Q_2$  denote an inner limiting set of  $I(0, 1)$  of measure zero containing  $P_1$ , and let  $P_2$  denote the first point of  $\alpha$  in  $M - Q_2$ . Let  $Q_3$  denote an inner limiting set of measure zero containing  $P_1 + P_2 + Q_2$ . Let  $P_3$  denote the first point of  $\alpha$  in  $M - Q_3, \dots$ . In general, if  $\bar{z}$  is an ordinal less than  $\omega_1$  and for each ordinal  $z, 1 < z < \bar{z}$ ,  $P_z$  and  $Q_z$  are defined, then  $\sum(P_z + Q_z), z < \bar{z}$ , is of measure zero and there exists an inner limiting set  $Q_{\bar{z}}$  of  $I(0, 1)$  of measure zero containing  $\sum(P_z + Q_z), z < \bar{z}$ . Now  $I(0, 1) - Q_{\bar{z}}$  is of power  $c$ . Let  $P_{\bar{z}}$  denote the first point of  $\alpha$  in  $I(0, 1) - Q_{\bar{z}}$ . Let  $\beta$  denote the sequence  $P_1, P_2, P_3, \dots, P_z, \dots$  and let  $M$  denote the subset of  $I(0, 1)$  whose points are the elements of  $\beta$ .

It is evident from the construction that  $M$  is of power  $\aleph_1$ . Suppose that  $K$  is a countable subset of  $M$ . Let  $P_{\bar{z}}$  denote the first point of  $M$  in  $\beta$  which follows  $K$  in  $\beta$  and let  $H = \sum P_z, z < \bar{z}$ . Then  $K$  is a subset of  $H$ , which is clearly an inner limiting set with respect to  $M$ . But  $H - K$  is countable since both  $H$  and  $K$  are countable, and therefore  $K$  is an inner limiting set with respect to  $M$ .

**AN EXAMPLE.** Let  $M'$  denote a subset of  $I(0, 1)$  of power  $\aleph_1$  such that every countable subset of  $M'$  is an inner limiting set with respect to  $M'$  and let  $Z'$  denote the set of all points  $(x, y)$  of the plane such that (1) both  $x$  and  $y$  are rational numbers, (2)  $0 < x < 1$ , and (3)  $y > 0$ . Furthermore, let  $\alpha$  denote a well ordered sequence of the points of  $M'$  such that if  $P$  is a point

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\* The symbolism  $G_{\delta}$  is used by some to denote an *inner limiting set*.

of  $M'$ , it belongs to  $\alpha$  and is preceded in  $\alpha$  by only a countable subset of  $M'$ . Let  $S$  denote a space consisting of the points of  $M'$  and  $Z'$  in which *sequential limit point* is defined as follows: I. A point  $P$  of  $Z'$  is the sequential limit point of a sequence of points  $P_1, P_2, P_3, \dots$  of  $S$  provided there exists a number  $N$  such that if  $n > N$ , then  $P_n = P$ . II. A point  $P$  of  $M'$  is the sequential limit point of a sequence of points  $P_1, P_2, P_3, \dots$  of  $Z'$  provided that (1) it is the sequential limit point of  $P_1, P_2, P_3, \dots$  in the plane and (2) the line  $PP_n$  approaches the normal to  $I(0, 1)$  at  $P$  as  $n$  increases without limit. III. A point  $P$  of  $M'$  is the sequential limit point of a sequence of points  $P_1, P_2, P_3, \dots$  of  $M'$  provided that for each element  $a$  of  $\alpha$  preceding  $P$  there exists an integer  $N$  such that if  $n > N$ , then either (1)  $P_n = P$  or (2)  $P_n$  is between  $a$  and  $P$  in  $\alpha$ ; IV. In general, a sequence of points  $P_1, P_2, P_3, \dots$  of  $S$  has a sequential limit point  $P$  provided that  $P$  is by I, II, and III the sequential limit point of each of its subsequences which lie in  $M'$  or  $Z'$ . In order that it may be easier to keep in mind which limit point notion is being used, we shall adopt the convention that if  $H$  is a subset of  $S$ ,  $H$  shall denote the point set as a subset of  $S$  and  $H'$  shall denote the corresponding subset of the plane.

From the definition of the above paragraph it is easy to see that  $S$  is a separable Fréchet space- $L$ . It will now be shown that  $S$  is normal.

Suppose that  $H$  and  $K$  are two mutually exclusive closed subsets of  $S$ . If both are uncountable, then  $H \cdot M$  and  $K \cdot M$  are uncountable, and letting  $A_1$  denote the first point of  $H$  in  $\alpha$ ,  $B_1$  denote the first point of  $K$  which follows  $A_1$  in  $\alpha$ ,  $A_2$  denote the first point of  $H$  which follows  $B_1$  in  $\alpha$ ,  $\dots$ , we see that  $A_1, A_2, A_3, \dots$  and  $B_1, B_2, B_3, \dots$  have the same sequential limit point. But since the sets are closed and mutually exclusive, this is impossible. Consequently one of the two sets is countable. We shall suppose that  $K$  denotes the countable set.

Since no point of  $Z$  is a sequential limit point of any sequence of distinct points of  $S$ , any subset of  $Z$  is a domain. Hence if  $H$  is a subset of  $Z$ ,  $D_H = H$  and  $D_K = S - H$  are two mutually exclusive domains containing  $H$  and  $K$  respectively. Likewise if  $K$  is a subset of  $Z$ , then  $D_H = S - K$  and  $D_K = K$  are two mutually exclusive domains containing  $H$  and  $K$  respectively.

On the other hand, if  $H$  and  $K$  contain points of  $M$ , then for

each point  $P$  of  $K \cdot M$ , let  $a$  denote the first point of  $\alpha$  which precedes  $P$  in  $\alpha$  such that no point of  $H \cdot M$  is between  $a$  and  $P$  in  $\alpha$ . Let  $d_P$  denote  $P$  together with all points of  $M$  which are between  $a$  and  $P$  in  $\alpha$ . Then  $d_P$  is a closed domain with respect to  $M$  containing  $P$ , and  $D_{1K} = \sum d_P$  is a domain with respect to  $M$  covering  $K \cdot M$  and containing no point of  $H$ . Furthermore,  $D_{1K}$  is closed, for if  $O$  were a limit point of  $\sum d_P$  not belonging to  $\sum d_P$ , then it would belong to  $M$  and be a limit point of the points  $P$  of  $K \cdot M$ , and hence belong to  $K \cdot M$ . Since  $D_{1K}$  is countable, let  $D_{1K} = O_1 + O_2 + O_3 + \dots$ . Then  $D'_{1K} = O'_1 + O'_2 + O'_3 + \dots$  is an inner limiting set with respect to  $M'$ . It shall be assumed for convenience that  $D'_{1K}$  does not contain the end points of  $I(0, 1)$ . There exists a sequence of point sets  $D'_1, D'_2, D'_3, \dots$  of  $I(0, 1)$  such that (1) for each  $n$ ,  $D'_n$  contains  $D'_{n+1}$ , (2) the common part of  $D'_1 \cdot M, D'_2 \cdot M, D'_3 \cdot M, \dots$  is  $D'_{1K}$ , and (3) for each  $n$ ,  $D'_n$  is the sum of a set of non-overlapping segments  $d'_{1n}, d'_{2n}, d'_{3n}, \dots$ . For each segment  $d'_{jn}$ , let  $r'_{jn}$  denote the interior of a regular hexagon in the plane having  $d'_{jn}$  as a diameter and let  $R'_n = \sum r'_{jn}$ . For each  $n$ , let  $t'_n$  denote the interior of an inverted equilateral triangle lying in  $R'_n$  with its base parallel to the  $X$ -axis and lower vertex at  $O'_n$ . Let  $T'$  denote the set of all points  $X'$  and  $Z'$  such that, for some  $n$ ,  $X'$  is in  $t'_n$ . Now  $D_K = D_{1K} + (T - T \cdot H)$  is a domain with respect to  $S$ . Furthermore, no point of  $M - D_{1K}$  is a limit point of  $D_K$ . For if  $P$  is a point of  $M - D_{1K}$ ,  $P$  is not a limit point of  $D_{1K}$  and there exists an integer  $k$  such that  $R'_k$  does not contain  $P'$ . But no sequence of points  $P'_1, P'_2, P'_3, \dots$  lying in  $T'$  has  $P'$  as a sequential limit point in the plane such that the line  $P'P'_n$  approaches the normal to the  $X$ -axis at  $P'$  since, for each  $n$ ,  $P'P'_n$  would make an angle of at least  $30^\circ$  with this normal when  $P'_n$  lies in  $R'_k$ . Hence  $D_K$  is closed and contains  $K$  but no points of  $H$ . Therefore,  $D_H = S - D_K$  and  $D_K$  are mutually exclusive domains containing  $H$  and  $K$  respectively, and  $S$  is normal.

The reader will observe that if  $N$  is an uncountable subset of  $S$ , then  $N \cdot M$  is uncountable and  $N$  has a limit point, namely, the first point  $P$  of  $\alpha$  such that infinitely many points of  $M$  precede  $P$  in  $\alpha$ . But it is clear that not every uncountable subset of  $S$  contains one of its limit points; for suppose that  $N$  is the set of all points  $P$  of  $M$  such that there is a first point of  $M$  in  $\alpha$  pre-

ceding  $P$  in  $\alpha$ . Then  $N$  is uncountable and contains none of its limit points.

In order to make an application of Theorem 4, two lemmas will be established. Throughout the rest of this paper  $M$  denotes a space satisfying Axiom 0 and parts 1, 2, and 3 of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* and is referred to as a *Moore space*  $M$ .

DEFINITION. A space is said to have the *Lindelöf property* provided that if  $G$  is a collection of domains of the space covering a point set  $K$ , then  $G$  contains a countable subcollection  $G'$  covering  $K$ .

LEMMA B. *In order that a Moore space  $M$  should have the Lindelöf property it is necessary and sufficient that every uncountable subset of  $M$  should have a limit point.\**

The necessity is well known. It remains only to establish the sufficiency.

PROOF. Suppose that  $G$  is a collection of domains covering a point set  $K$ . Let  $\alpha$  denote a well-ordering of  $K$ . For each  $n$ , let  $H_n$  denote a subcollection of  $G$  obtained by the following method. Let  $P_1$  denote the first element of  $\alpha$  such that some element  $g_1$  of  $G$  contains every region of  $G_n$  of Axiom 1 that contains  $P_1$ . Let  $P_2$  denote the first element of  $\alpha$ , not contained in  $g_1$ , such that some element  $g_2$  of  $G$  contains every region of  $G_n$  of Axiom 1 that contains  $P_2$ . In general, if  $\bar{z}$  is an ordinal and for each ordinal  $z$ ,  $z < \bar{z}$ ,  $P_z$  and  $g_z$  are chosen, then let  $P_{\bar{z}}$  denote the first point (if any) in  $\alpha$  not contained in  $\sum g_z$ ,  $z < \bar{z}$ , such that some element  $g_{\bar{z}}$  of  $G$  contains every region of  $G_n$  of Axiom 1 that contains  $P_{\bar{z}}$ . From this construction, it is clear that the set  $P_1, P_2, P_3, \dots, P_z, \dots$  has no limit point, for no region of  $G_n$  contains more than one of them. Hence  $H_n = g_1, g_2, g_3, \dots, g_z, \dots$  is a countable subcollection of  $G$ . Then  $G' = \sum_1^\infty H_n$  is a countable subcollection of  $G$ . Furthermore,  $G'$  covers  $K$ . For suppose that there is a point  $P$  of  $K$  not contained in any element of  $G'$ . Let  $g$  denote a domain of  $G$  containing  $P$ . By

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\* Lemma B is an advance over Theorem 18 on page 14 of R. L. Moore's *Foundations of Point Set Theory*. However, Moore's arguments may be used with some modifications to establish Lemma B.

Axiom 1 there exists a number  $n$  such that every region of  $G_n$  which contains  $P$  lies in  $g$ . Hence  $P$ , for some ordinal  $\bar{z}$ , is  $P_{\bar{z}}$  used in the selection of  $H_n$ , which is a contradiction.

LEMMA C. *If every uncountable subset of a Moore space  $M$  has a limit point,  $M$  is a completely separable metric space.*

PROOF. By Lemma B, for each  $n$ ,  $G_n$  of Axiom 1 contains a countable subcollection  $G'_n$  covering  $M$ ; hence  $G' = \sum G'_n$  is a countable collection of regions having the property that if  $P$  is a point of a region  $R$ , some element of  $G'$  contains  $P$  and lies in  $R$ . Hence  $M$  is completely separable. Professor Moore has pointed out that such a space is metric.\*

THEOREM 5. *If  $2^{\aleph_1} > 2^{\aleph_0}$ , then every separable normal Moore space  $M$  is completely separable and metric.*

This follows from Theorem 4 and Lemma C.

The author has tried for some time without success to prove that  $2^{\aleph_1} > 2^{\aleph_0}$ . But although Theorem 5 is unsatisfactory in this respect, it does raise a question of some interest: *Is every normal Moore space  $M$  metric?* This question is as yet unsettled. However, if the answer is *yes*, then it should be possible to establish directly certain results for normal Moore spaces  $M$  which are known to hold in metric spaces but which are known not to hold in all Moore spaces  $M$ . The author has established a number of such theorems but it seems likely that only one of them may be of use in settling the question itself.

THEOREM 6. *A normal Moore space  $M$  is completely normal.*

PROOF. Suppose that  $H$  and  $K$  are two mutually separate subsets of a normal Moore space  $M$ . For each integer  $n$ , let  $H_n$  denote the set of all points  $P$  of  $\bar{H}$  such that no region of  $G_n$  of Axiom 1 which contains  $P$  contains a point of  $\bar{K}$ . † Likewise, for each  $n$ , let  $K_n$  denote the set of all points  $P$  of  $\bar{K}$  such that no region of  $G_n$  which contains  $P$  contains a point of  $\bar{H}$ . For each  $n$ ,  $H_n$  and  $K_n$  are closed and  $H \subset \sum H_n$  and  $K \subset \sum K_n$ . Let  $D_{H_1}$  denote a domain containing  $H_1$  such that  $\overline{D_{H_1}} \cdot \bar{K} = 0$ . Let  $D_{K_1}$  denote a domain containing  $K_1$  such that  $\overline{D_{K_1}} \cdot (\overline{H + D_{H_1}}) = 0$ .

\* R. L. Moore, *Foundations of Point Set Theory*, pp. 459 and 464.

† The notation  $\bar{K}$  means  $K$  plus its limit points.

Let  $D_{H_2}$  denote a domain containing  $H_2$  such that  $\overline{D_{H_2}} \cdot \overline{(K + D_{K_1})} = 0$ . Let  $D_{K_2}$  denote a domain containing  $K_2$  and such that  $\overline{D_{K_2}} \cdot \overline{(H + D_{H_1} + D_{H_2})} = 0$ . This process may be continued and  $D_H = \sum D_{H_n}$  and  $D_K = \sum D_{K_n}$  are two mutually exclusive domains covering  $H$  and  $K$  respectively.

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## ON AN INTEGRAL EQUATION WITH AN ALMOST PERIODIC SOLUTION

BY B. LEWITAN

We assume that the function  $f(x)$  is almost periodic in the sense of H. Bohr and that the functions  $E(\alpha)$ ,  $\alpha E(\alpha)$  are absolutely integrable in  $[-\infty, \infty]$ .

**THEOREM.** *If all real zeros of the function*

$$\gamma(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(u) e^{-i\alpha u} du$$

*have integer multiplicities and only two limit points  $\infty$ ,  $\alpha^*$ , then every solution  $\phi(x)$  of the equation*

$$(1) \quad \int_{-\infty}^{\infty} E(\xi - x) \cdot \phi(\xi) d\xi = f(x)$$

*which is uniformly continuous and bounded in  $[-\infty, \infty]$  is almost periodic.*

**PROOF.** Without loss of generality we may assume that the finite limit point  $\alpha^*$  has the value 0; otherwise we multiply equation (1) by  $e^{-i\alpha^*x}$ .

Putting

$$f_n(x) = \frac{3}{2\pi} \int_{-\infty}^{\infty} f\left(x + \frac{2u}{n}\right) \frac{\sin^4 u}{u^4} du,$$

we obtain

$$\int_{-\infty}^{\infty} E(\xi) \phi_n(\xi + x) d\xi = f_n(x),$$