Let D_{H_2} denote a domain containing H_2 such that $\overline{D}_{H_2}\cdot\overline{(K+D_{K_1})}=0$. Let D_{K_2} denote a domain containing K_2 and such that $\overline{D}_{K_2}\cdot\overline{(H+D_{H_1}+D_{H_2})}=0$. This process may be continued and $D_H=\sum D_{H_n}$ and $D_K=\sum D_{K_n}$ are two mutually exclusive domains covering H and K respectively.

THE UNIVERSITY OF TEXAS

ON AN INTEGRAL EQUATION WITH AN ALMOST PERIODIC SOLUTION

BY B. LEWITAN

We assume that the function $f(x)$ is almost periodic in the sense of H. Bohr and that the functions $E(\alpha)$, $\alpha E(\alpha)$ are absolutely integrable in $[-\infty, \infty]$.

THEOREM. *If all real zeros of the function*

$$\gamma(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(u)e^{-i\alpha u}du$$

have integer multiplicities and only two limit points ∞, α^*, *then every solution* $\phi(x)$ *of the equation*

$$\int_{-\infty}^{\infty} E(\xi - x)\cdot\phi(\xi)d\xi = f(x) \tag{1}$$

which is uniformly continuous and bounded in $[-\infty, \infty]$ *is almost periodic.*

PROOF. Without loss of generality we may assume that the finite limit point α^* has the value 0; otherwise we multiply equation (1) by $e^{-i\alpha^* x}$.

Putting

$$f_n(x) = \frac{3}{2\pi}\int_{-\infty}^{\infty} f\left(x + \frac{2u}{n}\right)\frac{\sin^4 u}{u^4}\,du,$$

we obtain

$$\int_{-\infty}^{\infty} E(\xi)\phi_n(\xi + x)d\xi = f_n(x),$$

where

$$\phi_n(t) = \frac{3}{2\pi}\int_{-\infty}^{\infty}\phi\left(t + \frac{2u}{n}\right)\frac{\sin^4 u}{u^4}\,du.$$

If $v_n(\alpha)$ denotes the generalized Fourier transform of $\phi_n(t)$, then, in our case,† $v_n(\alpha)$ is a linear function for $\alpha > 2n$ and $\alpha < -2n$.

The functions $f_n(x)$ and $\phi_n(x)$ are differentiable and the derivative of $\phi_n(x)$ is bounded. The function $E(\xi)$ being absolutely integrable, we therefore obtain

$$\int_{-\infty}^{\infty} E(\xi)\phi_n'(\xi + x)d\xi = f_n'(x).$$

Putting

$$\lambda_\epsilon(\alpha) = \begin{cases} 1 & \text{for } |\alpha| \leqq \epsilon, \\ \left(2 - \frac{|\alpha|}{\epsilon}\right)^2\left(2\frac{|\alpha|}{\epsilon} - 1\right) & \text{for } \epsilon \leqq |\alpha| \leqq 2\epsilon, \\ 0 & \text{for } |\alpha| \geqq 2\epsilon, \end{cases}$$

and

$$\tau_\epsilon(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda_\epsilon(\alpha)e^{-i\alpha u}d\alpha,$$

$$f_{n,\epsilon}(x) = f_n'(x) - \int_{-\infty}^{\infty} f_n'(x + u)\tau_\epsilon(u)du,$$

$$\phi_{n,\epsilon}(x) = \phi_n'(x) - \int_{-\infty}^{\infty} \phi_n'(x + u)\tau_\epsilon(u)du,$$

we obviously have

$$\int_{-\infty}^{\infty} E(\xi)\cdot\phi_{n,\epsilon}(\xi + x)d\xi = f_{n,\epsilon}(x).$$

If $v_{n,\epsilon}(\alpha)$ and $u_{n,\epsilon}(\alpha)$ are generalized Fourier transforms of $\phi_{n,\epsilon}(x)$ and $f_{n,\epsilon}(x)$, then the relation‡

† S. Bochner, Mathematische Annalen, vol. 102 (1929), pp. 489–504, vol. 103 (1930), pp. 588–597.

‡ S. Bochner, loc. cit.

$$\gamma(\alpha)dv_{n,\epsilon}(\alpha) = du_{n,\epsilon}(\alpha)$$

holds.

It follows from the construction of the function $\lambda_\epsilon(\alpha)$ that the function $\gamma(\alpha)$ has a finite number of zeros in those intervals where $v_{n,\epsilon}(\alpha)$ is not linear. Consequently, by a result of S. Bochner,† the function $\phi_{n,\epsilon}(x)$ is almost periodic in the sense of H. Bohr.

When $\epsilon \to 0$, $\phi_{n,\epsilon}(x)$ converges to $\phi_n'(x)$ uniformly in $[-\infty, \infty]$. This follows from

$$\int_{-\infty}^{\infty} \phi_n'(x+u)\tau_\epsilon(u)du = \epsilon \int_{-\infty}^{\infty} \phi_n(x+u)\frac{\tau_\epsilon'(u)}{\epsilon}du \leqq \epsilon M \to 0,$$

where M is a constant. Hence, $\phi_n'(x)$ is almost periodic in the sense of H. Bohr. But $\phi_n(x)$ is bounded. Therefore, by the theorem of Bohr, $\phi_n(x)$ is also almost periodic. Finally, $\phi(x)$ being uniformly continuous, the sequence $\phi_n(x)$ converges to $\phi(x)$ uniformly in $[-\infty, \infty]$ as $n \to \infty$, and $\phi(x)$ is almost periodic itself.

We note that the assertion of the theorem remains valid if, more generally, the limit points of the zeros of $\gamma(\alpha)$ are isolated; it is also possible to drop the assumption that the zeros have integer multiplicities.

MATHEMATICAL INSTITUTE,
UNIVERSITY A. M. GORKI,
CHARKOW, U.S.S.R.

† S. Bochner, loc. cit.