AN INDECOMPOSABLE LIMIT SUM

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It is the object of this paper to investigate a certain simple monotone sequence of continua. The theorem of the paper states conditions under which the limit sum of the sequence is indecomposable. The precise formulation and proof of the theorem will be undertaken after the following lemma is established.

LEMMA. *Let K be a plane bounded indecomposable continuum and L a plane bounded continuum such that $K\cdot L\neq 0$, and that $c(L)$* includes a particular component λ containing the component δ of $c(L+K)$ with the following properties:*

(a) *the set L contains two distinct points, a and c, connected through δ by the arc B which divides δ into δ_i and δ_e, and λ into λ_i and λ_e;*

(b) *both λ_i and λ_e contain points of K.*

Then each component of $c(K+L)$ has as its boundary a proper subset of $K+L$.

The assumption that $c(K+L)$ has a component γ with boundary Γ such that $\Gamma\supset(K+L)$ will be shown contradictory. Let the boundaries of δ_i, δ_e, λ_i, λ_e be respectively Δ_i, Δ_e, Λ_i, and Λ_e. Suppose that δ is unbounded and also δ_e and λ_e, so that δ_i and λ_i will necessarily be bounded. Evidently $\lambda_i\supset\delta_i$ and $\lambda_e\supset\delta_e$. Consider first the case in which L is irreducible between a and c.

Both Λ_i and Λ_e contain L. For $\Lambda_i\subset L+B$ and $\Lambda_e\subset L+B$; so, since B is an arc with $L\cdot(B)\dagger=0$, $\Lambda_i\cdot L$ and $\Lambda_e\cdot L$ are continua containing $a+c$. If either of these is not identical with L, then L is reducible between a and c. The domains δ and γ are, moreover, identical, for both λ_i and λ_e contain points of K, therefore points of Γ, and therefore points of γ. There is thus an arc X in γ such that $X\cdot\lambda_i\neq 0$ and $X\cdot\lambda_e\neq 0$, and since $X\cdot L=0$, then $(B)\cdot X\neq 0$. This implies $X\cdot\delta\neq 0$, accordingly $\gamma\cdot\delta\neq 0$; and as both γ and δ are components of $c(K+L)$, then $\gamma=\delta$, and $\lambda+L\supset\Gamma\supset K$.

Let K_i be the sum of $K\cdot\lambda_i$ and of all the components of $L\cdot K$

* If X is a point set then $c(X)$ is the complement of X.

† If X is an arc then (X) is X with ends omitted.

containing limit points of $K\cdot\lambda_i$; and let K_e be a similar set constructed from λ_e. Each of these is a closed subset of K; and $K_i+K_e\supset K\cdot(\lambda_i+\lambda_e)\supset K\cdot\lambda$, so that $K_i+K_e\supset\overline{K\cdot\lambda}$. If $K_i+K_e\not\supset K\cdot L$, then $K\cdot L$ includes a component M such that $\overline{K\cdot\lambda}\cdot M=0$ and M may be enclosed in a simple closed curve C, not intersecting $K\cdot L$, and excluding $\overline{K\cdot\lambda}$. But $C\cdot(\overline{K\cdot\lambda}+K\cdot L)=0$ implies $C\cdot K=0$, because $\lambda+L\supset K$, so C separates K without intersecting it. Accordingly $K_i+K_e=K\cdot\lambda+K\cdot L=K$. But now if both K_i and K_e were continua, as they are proper subsets of K, the set $K=K_i+K_e$ would be decomposable. Thus either K_i or K_e is disconnected.

Let K_i be disconnected, that is, let $K_i=K_a+K_c$ where $K_a=\overline{K}_a$, $K_c=\overline{K}_c$, and $K_a\cdot K_c=0$. As $\lambda_i\cdot K$ is contained by Δ_i, the components of $K\cdot\lambda_i$ are an orderable collection and elements of this collection accessible from δ_i must belong to both K_a and K_c. Thus there is a pair of arcs B_a and B_c where $\delta_i\supset\{(B_a)+(B_c)\}$, $B_a\cdot K_a\cdot\lambda_i\neq 0$, $B_c\cdot K_c\cdot\lambda_i\neq 0$, $B_a\cdot(B)\neq 0$, $B_c\cdot(B)\neq 0$, and $B_a\cdot B_c=0$. Now B_a+B+B_c contains an arc D, such that $D\supset B_a+B_c$, $D\cdot B$ is a subarc of (B), $(D)\subset\delta$, and $\delta-(D)$ is a pair of domains δ_u and δ_b. Let the notation be so chosen that $\delta_b\subset\delta_i$ and $\delta_u\supset\delta_e$, the relationships being clear from the construction; and let the boundaries of δ_u and δ_b be Δ_u and Δ_b. Note that $\delta_b\cdot(a+c)=0$.

Now $\Delta_b\cdot(K+L)\not\subset K$. For $\Delta_b\supset D$ and consequently $\Delta_b\cdot K_a\neq 0$ and $\Delta_b\cdot K_c\neq 0$. But $\Delta_b\cdot(K+L)$ is a continuum because $\Delta_b\cdot c(K+L)=(D)$. As $K_a\cdot K_c=0$, $\overline{\Delta_b\cdot K_a}\cdot\overline{\Delta_b\cdot K_c}=0$; that is $\Delta_b\cdot(K+L)\not\subset\Delta_b\cdot(K_a+K_c)$. Therefore $\Delta_b\cdot L\cdot c(K_a+K_c)=L_k\neq 0$. Moreover L_k includes a component L_b such that $\overline{L_b}\cdot K_a\neq 0$ and $\overline{L_b}\cdot K_c\neq 0$, for otherwise it must follow again that $\Delta_b-(D)$ is disconnected. Now $K_e\not\supset L_b$, as otherwise there must be a component of $K\cdot L$ containing L_b and thus contained in both K_a and K_c. Thus $K_i+K_e\not\supset L_b$ and $L\cdot c(K)\supset h$, a point distinct from a and from c, and in Δ_b.

Let S be a circle with center h and radius such that $ce(S)$* is a set with no points in $K+D+B$, and such that in $e(S)\cdot\delta_e$ there is a ray R with end on (B). As L is irreducible between a and c, $L\cdot ci(S)$ consists of sets L_a'' and L_c'' mutually separated

* If S is a simple closed curve, its interior is $i(S)$ and its exterior is $e(S)$. Accordingly $ci(S)=S+e(S)$ and $ce(S)=S+i(S)$.

between a and c. Let L_a and L_c be the components of these containing a and c. Evidently $S \cdot L_a \neq 0$ and $S \cdot L_c \neq 0$. Let F be an arc such that $F \cdot S \neq 0$, $(F) \subset e(S)$, $(F) \subset \delta_b$, and $F \cdot (D \cdot B) \neq 0$. Let E be an arc such that $E \cdot S \neq 0$, $(E) \subset e(S)$, $(E) + E \cdot S \subset \lambda_e$, and $E \cdot B = F \cdot B$. Let G be an arc such that $(G) \subset i(S)$ and $G \cdot S = (E+F) \cdot S$. The set $E+F+G$ is a simple closed curve J, intersecting L only in $i(S)$.

The set $L_a + ce(S) + F + B$ bounds a bounded domain ϕ_a containing points of $K \cdot c(L) \cdot \lambda_i$; and $L_c + ce(S) + F + B$ bounds a similar domain ϕ_c. For $L_a + ce(S)$ is obviously a continuum and $B+F$ contains a cut F_a of its complement. Of the two components of $c(F_a + L_a + ce(S))$ which are bounded in part by (F_a) let ϕ_a be the bounded one. The boundary of ϕ_a contains either $B \cdot B_a$ or $B \cdot B_c$ (suppose the former), but not of course both. Thus by elementary reasoning from the constructions used, it appears that $\phi_a \supset B_a \cdot c(B)$ and thus contains a component of $K_a \cdot \lambda_i$. In a similar way ϕ_c may be proved to contain a component of $K_c \cdot \lambda_i$. Moreover $\phi_a \cdot \phi_c = 0$ and if $i(J) \supset \phi_a$ then $e(J) \supset \phi_c$ and vice versa. To be explicit, assume that $i(J) \supset \phi_a$.

That the domain λ_e contains an uncountably infinite number of components of $\lambda_e \cdot K$ each of which has a disconnected set of limit points in H will now be shown. Since $i(J) \supset \phi_a$ and $e(J) \supset \phi_c$, both $i(J)$ and $e(J)$ contain points of $K \cdot c(H)$ and thus contain points of every composant of K. Let $[Q_\alpha]$ be a collection of subcontinua of K, one and only one in each composant of K, and each one having both a point in ϕ_a and a point in ϕ_c. The elements of $[Q_\alpha]$ are uncountable and mutually exclusive. Any one, Q_g, of $[Q_\alpha]$ has a point in Φ_a and one in Φ_c, where Φ_a and Φ_c are the boundaries respectively of ϕ_a and ϕ_c; so $Q_g \cdot \Phi_a = Q_g \cdot (L_a + S + F_a) = Q_g \cdot L_a \neq 0$, and also $Q_g \cdot L_c \neq 0$. It appears indeed that Q_g has a point in each of the mutually separated closed sets $L \cdot e(J) \cdot ci(S)$ and $L \cdot i(J) \cdot ci(S)$, and as these two sets contain $L \cdot K$ they contain $L \cdot Q_g$. In consequence $Q_g \cdot c(L)$ has a component G_g with a limit point in $L \cdot i(J)$ and a limit point in $L \cdot e(J)$. As $L \cdot i(J)$ and $L \cdot e(J)$ are mutually separated, the limit set of G_g is disconnected since it includes no point of J; and also $G_g \cdot J = G_g \cdot (E+F+G) = G_g \cdot E \neq 0$, so $G_g \cdot \lambda_e \neq 0$ and therefore $\lambda_e \supset G_g$. Regard now the components $[K_\alpha]$ of $\lambda_e \cdot K$ which contain the members of $[G_\alpha]$. Each of these has limit points in both $i(J)$ and $e(J)$ and none in J, and so has a disconnected limit set in L. No two of these are identical, for no one, such as K_g, of them has $\overline{K}_g=K$ as it contains none of the points of K in $\phi_a+\phi_c$, and if K_g were to contain two of $[G_\alpha]$ then K_g would be a proper subcontinuum of K containing points of two different composants of K. Thus the collection $[K_\alpha]$ is one of the sort required.

Each set K_g+L is therefore a subcontinuum of $K+L$ separating the plane, a bounded component of its complement being δ_g. No pair of elements, δ_p and δ_q, of $[\delta_\alpha]$ can have a point in common unless one contains the other, for $K_p \cdot K_q=0$. Moreover if $\delta_p \supset \delta_q$ then $\delta_p \supset K_q$, and so $\delta_p \supset \gamma$, a contradiction as $\gamma \supset \delta$ was unbounded. Thus $[\delta_\alpha]$ is an uncountable collection of mutually exclusive domains in the plane, another contradiction establishing at last the lemma for this case.

None of the undiscussed suppositions made above requires any more justification than a suitable inversion of the plane except the assumption that L is irreducible between a and c. But if L is not irreducible between a and c, then it contains a subcontinuum W which is irreducible between a and c. By examining K and W it may be seen that the hypotheses of the lemma are fulfilled, so that the set $c(K+W)$ has no component with boundary $K+W$. Neither then does the less inclusive set $c(K+H)$.

THEOREM. *If $[D_i]$ is a simple infinite sequence of plane point sets such that, for each positive integer i, D_i is indecomposable and $D_i \subset D_{i+1}$, and such that the set $\overline{\sum_1^\infty D_i}$ is a plane bounded continuum Γ which is the frontier of γ, a component of its complement, then Γ is also indecomposable.*

The theorem is obvious if no more than a finite number of $[D_i]$ are distinct, as then Γ is identical with one of $[D_i]$. Assume accordingly that all of $[D_i]$ are different, other possible cases being not significant. Let $\sum_1^\infty D_i=D_m$ and $\Gamma-D_m=D_n$. Every point of D_n is a limit point of D_m, for $\Gamma=\overline{\sum_1^\infty D_i}=\overline{D_m}$ so $\overline{D_m}\supset D_n$. Every subcontinuum of D_n is a continuum of condensation of Γ, because when D_n contains the continuum K, then $\overline{D_m}\supset D_n\supset K$ implies $\overline{\Gamma-K}\supset\overline{\Gamma-D_n}\supset\overline{D_m}\supset K$. Moreover let d be a point of D_m. Now $D_{j+1}\supset D_j$ while $D_{j+1}-D_j\neq 0$, so D_j belongs to a single composant of D_{j+1}. Thus $\overline{D_{j+1}-D_j}\supset D_j$; that is $\overline{\Gamma - D_j} \supset \overline{D_{j+1} - D_j} \supset D_j$, or D_j is a closed set nowhere dense in the closed set Γ. But this being true for any value of j ($j=1, 2, 3, \cdots$) then $\sum_1^\infty D_j = D_m$ must be a set of the first category in the closed set Γ. Thus D_n is a set of the second category in Γ everywhere dense in the set Γ. That is, $\overline{D_n} \supset D_m$ so $\overline{D_n} \supset d$. Consequently every subcontinuum of D_m is a continuum of condensation of Γ, for when $D_m \supset K$, then $\overline{D_n} = \Gamma$ implies that $\overline{\Gamma - K} \supset \overline{\Gamma - D_m} \supset \overline{D_n} \supset K$. Henceforth consider γ unbounded.

The argument will be completed by showing that every proper subcontinuum of Γ is a continuum of condensation of Γ. Let K be such a continuum and, as the cases $D_m \supset K$ and $D_n \supset K$ have already been dealt with, suppose that $K \cdot D_m \neq 0$ and $K \cdot D_n \neq 0$. Clearly $K \not\supset D_m$ for $K \supset D_m$ would imply $K = \Gamma$. There must be some element of $[D_i]$ contained in part but not entirely by K. Let D_k be such an element. Thus, if $i > k$, the element D_i can not be a subset of K, for $K \supset D_i$ implies $K \supset D_i \supset D_k$.

Suppose that the set $\overline{D_k \cdot c(K)} = \Gamma_k$ is not connected. As $D_k + K \subset \Gamma$, $c(D_k + K) \supset c(\Gamma) \supset \gamma$. Thus there is a connected domain γ_k complementary to $D_k + K$ such that $\gamma_k \supset \gamma$. As $\bar{\gamma}_k \supset \bar{\gamma} \supset \Gamma$ then $\bar{\gamma}_k \supset D_k + K$. But $\bar{\gamma}_k = \gamma_k + B_k$, where B_k is the boundary of γ_k, and so $\bar{\gamma}_k \supset D_k + K$ implies $B_k \supset D_k + K$. Let G be a component of Γ_k containing the end of a ray R_g contained except for its end in γ_k. As every component of Γ_k consists of limit points of γ_k, and Γ_k is not connected, there is another component H of Γ_k containing the end of another ray R_h, ($R_g \cdot R_h = 0$), which is except for its end contained in γ_k.

Now there exists a simple closed curve C such that $i(C) \supset G$, $C \cdot \Gamma_k = 0$, $e(C) \supset H + R_h$, and $C \cdot R_g$ is a single point. Upon tracing C in opposite directions from $C \cdot R_g$, first points of $D_k + K$ are clearly encountered. Let the subarc of C thus identified be B. But $(B) \cdot (D_k + K) = 0$ by selection, and $B \cdot (D_k + K) \subset B \cdot (\Gamma_k + K)$ $\subset C \cdot \Gamma_k + B \cdot K \subset B \cdot K \subset K$, so B is a cut of the unbounded complementary domain γ_d of K. Thus $\gamma_d - (B)$ consists of two domains, a bounded one γ_b and an unbounded one γ_u. As $R_h \cdot (B + K) \subset R_h \cdot (C + K) \subset R_h \cdot C + R_h \cdot K = 0$, then $\gamma_u \supset R_h$, and thus $\gamma_u \cdot D_k \neq 0$, for indeed $\gamma_u \supset H$. Upon considering γ_b it may be seen with reasonable ease that the single point $B \cdot R_g$ separates R_g into two parts, the unbounded one of which is a subset of γ_u whereas the bounded one is a subset of γ_b. As the end of the bounded part is in G, then γ_b also contains points of Γ_k as it contains G. These facts make it clear also that the ends of B are distinct, for if identical they would coincide with a cut point of the continuum D_k, although D_k is indecomposable.

A contradiction of the lemma now appears, for this is the situation: the plane continuum D_k is indecomposable and bounded and K is a bounded continuum such that $K \cdot D_k \neq 0$, the set K contains two distinct points connected in $c(D_k+K)$ by an arc B having only its ends in common with D_k+K and separating the component of $c(K)$ which contains it into two domains γ_b and γ_u, both γ_b and γ_u contain points of D_k, and there is a component of $c(D_k+K)$ whose boundary is identical with D_k+K. As this is ridiculous, the set Γ_k is connected as was to be proved.

But if K is not a continuum of condensation of Γ, there exists a point s of K and a circle S such that $i(S) \supset s$ and $e(S) \supset \overline{\Gamma - K}$. But $\overline{D_m} = \Gamma$ so $i(S) \cdot D_m \neq 0$; that is, there exists a subscript q such that $i(S) \cdot D_q \neq 0$. For any subscript $j > q$, then $i(S) \cdot D_j \neq 0$ as $i(S) \cdot D_j \supset i(S) \cdot D_q \neq 0$. Let r be a natural number greater than k and greater than q. Then $\overline{D_r \cdot c(K)} = \Gamma_r$ is non-vacuous and connected as has been seen already. But $\Gamma_r \cdot i(S) \subset \overline{\Gamma - K} \cdot i(S) = 0$, so Γ_r fails to contain any point of the non-vacuous set $i(S) \cdot D_r$. Therefore Γ_r is a proper subcontinuum of D_r, and must accordingly belong to a single composant D_r^a of D_r. As $D_r \cdot c(K) \neq 0$ by supposition, there are points of D_r not in K. But let D_r^b be a second composant of D_r. As $D_r^a \supset \Gamma_r$, then $D_r^b \subset K$. Accordingly $\overline{D_r^b} \subset K$, and as $\overline{D_r^b} = D_r$, finally $D_r \subset K$, a contradiction.

As the contradiction is now general, the theorem is proved.

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