A NOTE ON THE CESÀRO METHOD OF SUMMATION*

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1. *Introduction.* A sequence $\{S_n\}$, or a series $\sum U_n$ with partial sums S_n, is said to be summable by the Cesàro mean of order α, or summable (C, α), to the sum s, if $\sigma_n^\alpha = S_n^\alpha / A_n^\alpha \to s$,† where S_n^α and A_n^α are given by the following relations:

$$(1)\quad (1-x)^{-\alpha-1} = \sum A_n^\alpha x^n; \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!};$$

$$(2)\qquad \sum S_n^\alpha x^n = (1-x)^{-\alpha} \sum S_n x^n = (1-x)^{-\alpha-1} \sum U_n x^n;$$

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} U_\nu;$$

and where α is any complex number other than a negative integer.‡ We shall restrict ourselves in this note to real orders of summability. It is known that if a sequence or series S is summable (C, α), $\alpha > -1$, it is summable (C, α'), $\alpha' > \alpha$, to the same sum.§ If a sequence or series S is summable (C, α) for all $\alpha \geqq \gamma$, then the lower limit of all such possible values of γ is called by Chapman‖ the *index of summability* of S.

It is sometimes easier to find the indices of summability and the sums of certain subsequences of a sequence S than to find the index and sum of S itself. As a trivial example, let $\{S_n\}$ be the sequence of partial sums of Leibniz's series $1-1+1-1+\cdots$. Then $S_{2k}=1$, $S_{2k+1}=0$, and it is easily seen that $\{S_{2k}\}$ is summable to the value 1 and $\{S_{2k+1}\}$ to the value 0 by the Cesàro mean of any order. It is the purpose of this note

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† Superscripts will not denote exponents when applied to capital letters and to the letter σ.

‡ For a systematic account of the Cesàro method, see Kogbetliantz, *Sommation des Séries et Intégrales Divergentes par les Moyennes Arithmétiques et Typiques*, Paris, 1931.

§ Kogbetliantz, op. cit., p. 17.

‖ Proceedings of the London Mathematical Society, (2), vol. 9 (1911), pp. 369–409; p. 378.

to investigate the relation between the Cesàro sum of a sequence and the Cesàro sums of equally spaced subsequences. We shall also generalize a remark of Chapman concerning diluted series.

2. *The Sum in Terms of the Sums of Subsequences.* Let

$$\sigma_{H,n}^{\alpha} = \frac{\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_{\lambda\nu+H}}{A_n^{\alpha}}, \tag{3}$$

where λ is any positive integer and $0 \leqq H \leqq \lambda - 1$.

THEOREM I. *If for some* $\alpha > 0$, $\sigma_{H,n}^{\alpha} \to s_H$, $H = 0, 1, \cdots, \lambda - 1$ (*uniformly with respect to any set of parameters* (y) *on which the terms of the sequence* $\{S_n\}$ *may depend*), *then* $\sigma_n^{\alpha} \to (\sum_{H=0}^{\lambda-1} s_H)/\lambda$ (*uniformly in* (y)).

PROOF. Let K be any integer such that $0 \leqq K \leqq \lambda - 1$, and let $A_k^{\delta} = 0$ for $k < 0$. Then

$$\begin{aligned} \sigma_{\lambda n+K}^{\alpha} &= \frac{1}{A_{\lambda n+K}^{\alpha}} \sum_{\nu=0}^{\lambda n+K} A_{\lambda n+K-\nu}^{\alpha-1} S_\nu \\ &= \frac{1}{A_{\lambda n+K}^{\alpha}} \sum_{H=0}^{\lambda-1} \sum_{\nu=0}^{n} A_{\lambda n+K-(\lambda\nu+H)}^{\alpha-1} S_{\lambda\nu+H} \\ &= \frac{1}{\lambda} \sum_{H=0}^{\lambda-1} \tau_{H,L,n}, \end{aligned}$$

where $L = K - H$ and

$$\tau_n = \tau_{H,L,n} = \frac{\lambda}{A_{\lambda n+K}^{\alpha}} \sum_{\nu=0}^{n} A_{\lambda(n-\nu)+L}^{\alpha-1} S_{\lambda\nu+H}. \tag{4}$$

We shall prove that $\tau_n \to s_H$ uniformly in (y), which will clearly suffice to prove the theorem. Our procedure is essentially this: Let C_α^{-1} be the inverse of the matrix of the transformation (3) and let Γ be the matrix of the transformation (4). We show that ΓC_α^{-1} satisfies the Silverman-Toeplitz regularity conditions.*

* See, for instance, Carmichael, this Bulletin, vol. 25 (1918), pp. 97–131; p. 109. The method of proof used in this note is discussed in the same reference on pp. 112–113. The author is indebted to R. P. Agnew for the suggestion that the procedure be explained in terms of matrices.

We shall arrange the work in such a way that the order of magnitude of the quantity $\tau_n - s_H$ will be exhibited.

In the sequel, A_n^α will be defined by the first formula of (1) for all values of α, including negative integers. By means of (2), the inverse of the transformation (3) is easily found to be

$$S_{\lambda\nu+H} = \sum_{k=0}^{\nu} A_{\nu-k}^{-\alpha-1} A_k^\alpha \sigma_{H,k}^\alpha.$$

If we replace $\sigma_{H,n}^\alpha$ by $s_H + \epsilon_{H,n}$, substitute in (4), and rearrange terms, we have

$$\tau_n = \frac{s_H \sum_{\nu=0}^{n} \lambda A_{\lambda(n-\nu)+L}^{\alpha-1}}{A_{\lambda n+K}^\alpha} + \frac{\lambda \sum_{\nu=0}^{n} \epsilon_{H,\nu} A_\nu^\alpha B_{\lambda(n-\nu)+L}}{A_{\lambda n+K}^\alpha} = \Sigma_1 + \Sigma_2,$$

where B_n is given by the relation $\sum B_n x^n = (1-x^\lambda)^\alpha (1-x)^{-\alpha}$.

Now let $\Delta_n = \Sigma_1 - s_H$. Then

$$\Delta_n = s_H \frac{\sum_{k=0}^{\lambda-1} \sum_{\nu=0}^{n-1} (A_{\lambda\nu+L}^{\alpha-1} - A_{\lambda\nu+k}^{\alpha-1}) + \lambda A_{\lambda n+L}^{\alpha-1} - \sum_{k=0}^{L} A_{\lambda n+k}^{\alpha-1}}{A_{\lambda n+K}^\alpha}.$$

Letting $\delta_n = \sum_{\nu=0}^{n-1}(A_{\lambda\nu+L}^{\alpha-1} - A_{\lambda\nu+k}^{\alpha-1})$, we prove that $\delta_n = O(1) + O(n^{\alpha-1})$. The result is immediate if $\alpha = 1$. Suppose $\alpha \neq 1$, and $L \geqq k$; the proof is similar if $L < k$. We have

$$(5) \qquad \delta_n = \sum_{\nu=0}^{n-1} \sum_{h=0}^{L-k-1} (A_{\lambda\nu+L-h}^{\alpha-1} - A_{\lambda\nu+L-h-1}^{\alpha-1}) = \sum_{\nu=0}^{n-1} \sum_{h=0}^{L-k-1} A_{\lambda\nu+L-h}^{\alpha-2}.$$

We make use of the following lemma of Andersen:*

$$\sum_{\nu=0}^{n} | A_\nu^{-\beta} A_{n-\nu}^{-\gamma} | = O(n^{-\beta}) + O(n^{-\gamma}) + O(n^{-\beta-\nu+1}),$$

provided that $-\beta$, $-\gamma$, and $-\beta-\gamma+1$ are not negative integers. If $-\beta-\gamma+1$ is a negative integer, the equation reads as follows:

$$\sum_{\nu=0}^{n} | A_\nu^{-\beta} A_{n-\nu}^{-\gamma} | = O(n^{-\beta}) + O(n^{-\gamma}).$$

* Andersen, *Studier over Cesàros Summabilitetsmetode*, København, 1922, pp. 22–23.

When the lemma is applied to (5), our assertion concerning the order of δ_n is established at once.

It follows (see (7) below) that

$$\Delta_n = O(n^{-\alpha}) + O(n^{-1}). \tag{6}$$

Equation (6) expresses the fact that the sums of the rows of matrix ΓC_α^{-1} converge to unity, which, of course, is one of the three Silverman-Toeplitz regularity conditions.

Turning to Σ_2, we first observe that $|B_n| \leqq M_1 |A_n^{-\alpha-1}|$, where M_1 is independent of n. The remark is obvious if $\lambda = 2$, or if α is an integer; and in the general case, the reader will have no difficulty in supplying an induction proof based on Andersen's lemma and on the fact that $\sum B_n x^n = \prod_{p=1}^{\lambda-1}(1-a_p x)^\alpha$, where $|a_p| = 1$. Our second observation is that the well known relation

$$A_n^\delta \cong n^\delta / \Gamma(\delta + 1), \quad \delta \neq -1, -2, \cdots, \tag{7}$$

gives us the inequality $|A_{\lambda n}^\delta| \leqq M_2 |A_n^\delta|$, M_2 independent of n, which is valid for all real values of δ. From these remarks it follows that

$$\begin{aligned} |\Sigma_2| &\leqq M_1 \frac{\sum_{\nu=0}^{n} |\epsilon_{H,\nu} A_\nu^\alpha A_{\lambda(n-\nu)+L}^{-\alpha-1}|}{A_{\lambda n+K}^\alpha} \\ &\leqq M_1 M_2 \frac{\sum_{\nu=0}^{n} |\epsilon_{H,\nu} A_\nu^\alpha A_{n-\nu}^{-\alpha-1}|}{A_{\lambda n+K}^\alpha}. \end{aligned}$$

Now

$$\frac{A_\nu^\alpha A_{n-\nu}^{-\alpha-1}}{A_{\lambda n+K}^\alpha} = o(1) \tag{8}$$

for any fixed value of ν, by (7), and

$$\frac{\sum_{\nu=0}^{n} |A_\nu^\alpha A_{n-\nu}^{-\alpha-1}|}{A_{n+K}^\alpha} = O(1) \tag{9}$$

by (7) and by the lemma of Andersen if α is fractional, otherwise because $A_n^{-\alpha-1} = 0$ for sufficiently large values of n. It follows as in the proof of the Silverman-Toeplitz Theorem that if $\epsilon_{H,n}\to 0$ uniformly in (y), Σ_2 behaves likewise. Equations (8) and (9) express the fact that the matrix ΓC_α^{-1} satisfies the remaining two regularity conditions.

3. *Degree of Convergence.* The proof of the theorem is now complete. To summarize, we have shown that whether or not the numbers $\epsilon_{H,n}\to 0$, we have

$$(10)\quad \left| \sigma_{\lambda n+K}^{\alpha} - s \right| \leqq M\left(n^{-\alpha} + n^{-1} + n^{-\alpha} \sum_{H=0}^{\lambda-1} \sum_{\nu=0}^{n} \left| \epsilon_{H,\nu} A_\nu^{\alpha} A_{n-\nu}^{-\alpha-1} \right| \right),$$

$(K=0, 1, \cdots, \lambda-1)$, where M is independent of n and K, and $s=(\sum_{H=0}^{\lambda-1} s_H)/\lambda$; and further, that if $\epsilon_{H,n}\to 0$ uniformly in (y), $(H=0, 1, \cdots, \lambda-1)$, then $\sigma_n^\alpha \to s$ uniformly in (y).

By considering the sequence 1, 0, 1, 0, $\cdots$, the reader will have no trouble in showing that the first two terms of our estimate of $\left|\sigma_{\lambda n+k}^{\alpha}-s\right|$ in (10) cannot be improved. If certain restrictions are placed on the sequences $\{\epsilon_{H,n}\}$, it can be proved that the third term in the right hand member of (10) is $O(\epsilon_n)$. We shall amplify this statement only by mentioning two special cases:

(a) If $\sigma_{H,n}^{\alpha}-s_H=O(n^{-\delta})$, $(H=0, 1, \cdots, \lambda-1)$, uniformly in (y), then $\sigma_n^\alpha - s = O(n^{-1})+O(n^{-\alpha})+O(n^{-\delta})$ uniformly in (y).
(b) If $\sigma_{H,n}^{\alpha}-s_H=O(n^{-\alpha}\log n)$, $(H=0, 1, \cdots, \lambda-1)$, uniformly in (y), then $\sigma_n^\alpha - s = O(n^{-1})+O(n^{-\alpha}\log n)$.*

4. *The Sum in Terms of the Sums of Subseries.* Chapman called a series "uniformly diluted" if between every pair of terms of the series is placed a constant number of zero terms.† His conjecture‡ that uniform dilution can affect neither the sum nor the summability of a series is a special case of the following theorem.

THEOREM II. *If* $\sum U_{\lambda n+H}$ *is summable* (C, α) *to the sum* u_H, $(H=0, 1, \cdots, \lambda-1)$, *for some* $\alpha > -1$ *(uniformly with respect*

* The author encountered the problems solved in this note while generalizing certain results concerning the Taylor series on the circle of convergence. The importance of the two special cases considered here is due to Szász's work on the degree of summability of the Fourier series, Acta Szeged, vol. 3 (1927), pp. 38–48.

† Loc. cit., p. 404.

‡ Ibid.

to any set of parameters (y) *on which the terms of the series* $\sum U_n$ *may depend*), *then* $\sum U_n$ *is summable* (C, α) *to the value* $\sum_{H=0}^{\lambda-1} u_H$ (*uniformly in* (y)).

The close connection between this theorem and Theorem I is apparent when we notice that

$$\sigma_{H,n}^{\alpha} = \left(\sum_{h=0}^{H} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} U_{\lambda\nu+h} + \sum_{h=H+1}^{\lambda-1} \sum_{\nu=0}^{n-1} A_{n-\nu}^{\alpha} U_{\lambda\nu+h} \right) / A_n^{\alpha}.$$

Since the conclusion of Theorem I is not true for $\alpha \leqq 0$ (see §5 below), a general proof is more conveniently given by the methods of §2 than by reference to this theorem. Equation (3) is to be replaced by the transformation $\sigma_{H,n}^{\alpha} = (\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} U_{\lambda\nu+H})/A_n^{\alpha}$ and (4) by the transformation $\tau_n' = (\sum_{\nu=0}^{n} A_{\lambda(n-\nu)+L}^{\alpha} U_{\lambda\nu+H})/A_{\lambda n+k}^{\alpha}$. The reader will have no difficulty in filling in the details.

5. *Discussion.* Of course the converse of neither Theorem I nor Theorem II is true. In certain senses the theorems cannot be strengthened. In the first place, the index of summability of a sequence or series may be equal to the greatest of the indices of the subsequences or subseries. An example is the series $1+0-2+0+3+0-4+0+ \cdots$, whose index is 1. (We remark furthermore that this series is not summable $(C, 1)$.) Again, the conclusion of Theorem I is false for $\alpha < 0$, even if the sums s_H be equal, as the reader may show by examining the sequence 0, A_0^{δ}, 0, A_1^{δ}, 0, A_2^{δ}, $\cdots$, where $-1 < \alpha < \delta < 0$. Finally, the conclusion of Theorem II is false for $\alpha < -1$; an example is the series $0+A_0^{\delta}+0+A_1^{\delta}+0+A_2^{\delta}+ \cdots$, where $-2 < \alpha < \delta < -1$.

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