NOTE ON A THEOREM CHARACTERIZING GEODESIC ARCS IN COMPLETE, CONVEX METRIC SPACES

BY L. M. BLUMENTHAL

1. *Introduction.* In his four *Untersuchungen über allgemeine Metrik*,* Menger initiated the systematic study of the metric geometry of abstract semi-metric and metric spaces. Among the most important of the notions Menger introduced in such spaces is that of *convexity* which leads, in complete metric spaces, to the existence of geodesic arcs joining each pair of points a, b of the space. Such an arc is congruent to a line segment of length ab.†

Concerning geodesic arcs in complete, convex metric spaces, Menger gives the following theorem:‡

THEOREM. *The geodesic arcs joining two points a, b of a complete, convex metric space are characterized among all arcs joining a, b by the following property: if p, q are elements of a geodesic arc joining a, b (p, q both distinct from a, b) then either p is between§ a and q, or p is between q and b, or p is identical with q.*

That a geodesic arc joining a, b has this property follows directly, as Menger observes, from the fact that such an arc may be imbedded congruently in a line segment of length ab. To show, however, that the property is *characteristic* for geodesic arcs it must be shown, of course, that every arc joining a, b that has this property is a geodesic arc. This sufficiency of the property is not shown by Menger (two proofs of the *necessity* of the condition being given instead). As the theorem is of use in developing some of the properties of convex spaces, and as a search of the literature, as well as conversation with Menger, has re-

* Mathematische Annalen, vol. 100 (1928), pp. 75–163; vol. 103 (1930), pp. 466–501.

† The term "geodesic arc" is used by the author in the geometry of distances always in the sense of minimizing geodesic.

‡ *Erste Untersuchung*, loc. cit., p. 91.

§ A point q lies between two points p, r if and only if $p \neq q \neq r$ and $pq+qr=pr$. We symbolize this relation by writing pqr.

vealed that apparently no previous notice of this lacuna has occurred, it seems worthwhile to fill the indicated gap by supplying a short proof of the sufficiency of the property.

2. *The Theorem.* We wish to prove the following theorem:

THEOREM. *If B is an arc joining two points a, b of a complete, convex metric space, and if B has the property that for every pair of its points p, q (distinct from a, b) either apq or qpb or $p=q$ exists, then B is a geodesic arc.*

PROOF. We assume that B is not a geodesic arc and deduce a contradiction.

From the assumption that B is not a geodesic arc it follows that B contains at least one pair of points without containing a middle point of the pair.* For if this were not the case, then B, being a closed, compact subset of a metric space, and containing a middle point for each pair of its elements, would, by the Existence Theorem for geodesic arcs,† contain a geodesic arc B' joining a, b. But since B is itself an arc with endpoints a, b, it is clear that B would be identical with B', which contradicts our assumption that B is not a geodesic arc.

Further, it is easy to show that B contains at least one pair of points b_0, b_1, *each distinct from* a, b, without containing a middle point of the pair; for a denial of this assertion leads at once (because of the continuity of the metric and the compactness of B) to the conclusion that B contains a middle point for each pair of its elements, which, as we have seen, violates the assumption that B is not a geodesic arc.

Though B does not contain a middle point of b_0, b_1, that subarc of B joining b_0, b_1 does contain a point c which is equidistant from b_0 and b_1; that is, c satisfies the relations

$$(1) \qquad b_0c = cb_1 > 0; \qquad b_0c + cb_1 > b_0b_1.$$

It follows that the three points b_0, c, b_1 are not linear.‡

Apply, now, the property stated in the theorem to each of

* A point q is a middle point of two points p, r if and only if q is between p and r, and q is equidistant from p, r.

† *Erste Untersuchung*, loc. cit., p. 89.

‡ Three pairwise distinct points are linear if and only if one of the points is between the other two.

the pairs of *distinct* points b_0, b_1; c, b_0; b_1, c in turn. We obtain the result that ab_0b_1 or b_1b_0b *and* acb_0 or b_0cb *and* ab_1c or cb_1b exist. But it is readily verified that no one of the eight possible combinations can exist, for assuming any one of them leads, by an application of the transitive property of the betweenness relation in metric spaces, to the conclusion that the triple b_0, c, b_1 is linear, in violation of the relations (1) above.* This completes the proof of the theorem.

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A TRANSFORMATION ASSOCIATED WITH THE TRISECANTS OF A RATIONAL TWISTED QUINTIC CURVE†

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1. *Introduction.* This transformation is generated by the use of a (1, 1) correspondence between a pencil of ruled cubic surfaces $|F|$, and their simple directrices which are the trisecants of a rational twisted quintic curve C_5. All of the cubic surfaces contain C_5 and have the quadrisecant of C_5 as a double line l. Through a general point P of space passes one F whose simple directrix r determines with P a plane tangent to the ruled surface of trisecants of C_5 at a point Q on r. The line PQ meets F in a residual point P' which is the image of P in an involutorial Cremona transformation of order 43. A special feature of the transformation is the existence of two ruled surfaces whose generators are parasitic lines of the transformation. One of these surfaces is a principal surface, and the other is not.

Other transformations generated by a somewhat similar method have been discussed by the author in recent papers.‡

2. *The Pencil of Cubic Surfaces.* The equation of a pencil of cubic surfaces, parameter λ, with a double line $l \equiv x_1 = x_2 = 0$, is

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* *Erste Untersuchung*, loc. cit., p. 78. For example, the combination ab_0b_1, acb_0, ab_1c cannot exist since the second and third of these relations imply the existence of b_1cb_0.

† Presented to the Society, December 29, 1936.

‡ This Bulletin, vol. 42 (1936), pp. 535–540. Tôhoku Mathematical Journal, vol. 43, (1937), pp. 174–177.