ON SYMMETRIC DETERMINANTS

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In a former paper* the writer proved the following theorem:

THEOREM A. *If* $D=|a_{ij}|$ *is a symmetric determinant of order* $n>4$ *with* a_{ij} *real and* $a_{ii}=0$, $(i=1, 2, \cdots, n)$, *and if all fourth-order principal minors of* D *are zero, then* D *vanishes.*

The purpose of this note is to give some results which are obtained immediately from this theorem and which are in one sense a generalization of this theorem.

Suppose D is a symmetric determinant of order $n>4$, with *real* elements, in which all principal minors of order $n-1$ and also all principal minors of order $n-4$ are zero. If $D'=|A_{ij}|$ is the adjoint of D, then $A_{ii}=0$, $(i=1, 2, \cdots, n)$. Each fourth-order principal minor of D' is equal to the product of D^3 by a principal minor of D of order $n-4$.† Therefore D' satisfies the conditions of Theorem A and hence is zero. But $D'=D^{n-1}$ and hence D is also zero and we have the following theorem:

THEOREM 1. *If* D *is a symmetric determinant of order* $n>4$, *with real elements, in which all principal minors of order* $n-1$ *and also all principal minors of order* $n-4$ *are zero, then* D *vanishes.*

Suppose D is a symmetric determinant of order $n>4$, with *real* elements, in which all principal minors of some order $k>3$ and also all principal minors of order $k-3$ are zero. Let M be any $(k+1)$-rowed principal minor of D, $(M=D$ if $n=5)$, then M is a determinant satisfying the conditions of Theorem 1 and hence M is zero. Therefore, in D, all principal minors of order k and also all principal minors of order $k+1$ are zero, hence D is of rank $k-1$ or less.‡ We have thus proved the following theorem:

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* *On real symmetric determinants whose principal diagonal elements are zero*, this Bulletin, vol. 38 (1932), pp. 259–262. See also, *On symmetric determinants*, American Mathematical Monthly, vol. 41 (1934), pp. 174–178.

† Bôcher, *Introduction to Higher Algebra*, p. 31.

‡ Bôcher, loc. cit., page 57, Theorem 2.

THEOREM 2. *If D is a symmetric determinant of order $n>4$, with real elements, in which all principal minors of some order $k>3$ and also all principal minors of order $k-3$ are zero, then D is of rank $k-1$ or less.*

If $n>5$, and $k<n-1$ the minors of Theorem 2 may be divided into two complementary sets such that if all minors of either set are zero the determinant vanishes. This division into sets may be done in n different ways.

Suppose D is a symmetric determinant of order $n>5$, with *real* elements, and M is a principal minor of D of order $n-1$. If all principal minors of M of some order $k>3$ and also all principal minors of M of order $k-3$ are zero, then M is of rank $k-1$ or less by Theorem 2. Let us suppose now that M is in the upper left hand corner of D and expand D according to the products of the elements of the last row and the last column. We get

$$D = a_{nn}M - \sum_{i,j=1}^{n-1} a_{ni}a_{jn}\alpha_{ij},$$

where α_{ij} is the cofactor of a_{ij} in M. If now we make the further restriction that k be less than $n-1$, then, since the rank of M is $k-1$ or less, each $\alpha_{ij}=0$ and consequently $D=0$. We have, therefore, the following result:

THEOREM 3. *If D is a symmetric determinant of order $n>5$, with real elements, and M is a principal minor of D of order $n-1$, and if all principal minors of M of some order k, $3<k<n-1$, and also all principal minors of M of order $k-3$ are zero, then D vanishes.*

Suppose D is a symmetric determinant of order $n>5$, with *real* elements, and that M is a principal minor of D of order $n-1$. Suppose also that all principal minors of D of some order $n-t$ and also all principal minors of D of order $n-t+3$, $(t>3)$, which are not minors of M, are zero. We may assume further, without loss of generality, that M is in the upper left hand corner of D. Let D' be the adjoint of D and M' be the minor of D' corresponding to M in D. Any principal minor of M' of order t (of order $t-3$) is equal to the product of D^{t-1} (D^{t-4}) by the complement in D of the corresponding minor in M. This complementary minor is a minor of D of order $n-t$ $(n-t+3)$ and is not a minor of M and hence is zero by hypothesis. Therefore M' is a symmetric determinant of order $n-1>4$, with *real* elements, in which all principal minors of some order $t>3$ and also all principal minors of order $t-3$ are zero, and hence M' is of rank $t-1$ or less by Theorem 2. If we make the further restric tion that t be less than $n-1$ we find, by expanding D' according to the products of the elements of the last row and the last column, that D' is zero. But $D'=D^{n-1}$ and hence D is zero also.

If we write $n-t+3=k$, since $3<t<n-1$, we have $4<k<n$ and hence the truth of the following theorem is apparent:

THEOREM 4. *If D is a symmetric determinant of order $n>5$, with real elements, and M is any principal minor of D of order $n-1$, and if all principal minors of D of some order $k>4$ and also all principal minors of D of order $k-3$, which are not minors of M, are zero, then D vanishes.*

In a second paper the writer* proved a theorem stated as follows:

THEOREM B. *If $D=|a_{ij}|$ is a symmetric determinant of order $n>5$, in which $a_{ii}=0$, $(i=1, 2, \cdots, n)$, and M is any principal minor of D of order $n-1$, then if all fourth order principal minors of D which are not minors of M are zero, D vanishes.*

From this theorem we see that the restriction that the elements of D be real is not necessary in Theorem A when n is greater than five. Consequently the theorems of this paper may be extended to include determinants with complex elements. Theorem 1 is true for complex elements if $n>5$. Theorem 2 is true for complex elements if $n>5$ and $k>4$. Theorems 3 and 4 are true for complex elements if $n>6$ and $4<k<n-1$.

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* *A theorem on symmetric determinants*, this Bulletin, vol. 38 (1932), pp. 545–550.