SOME SYMBOLIC IDENTITIES*

BY L. I. NEIKIRK

Differential equations were first solved by symbolic methods in England and on the continent in the first half of the last century. The differentiation symbol was treated as a symbol of quantity with restrictions. Then followed symbolic treatment of invariants and covariants, Cayley's hyperdeterminant, and Aronhold's symbolic notations. These were followed by Bli­ssard's umbral notation in the theory of numbers.

This paper is devoted to showing that these are all reducible to symbolic differentiation.

If we represent differentiation by the symbol $D$ and separate the symbols of operation from the symbols of quantity, then any analytic identity, such as $\Phi_1(y) = \Phi_2(y)$, will give an operational identity, $\Phi_1(D) = \Phi_2(D)$.

If this operational identity is applied to a second identity

$$F_1(x) = F_2(x)$$

the result will be a new identity. Most identities obtained in this way are easily obtained otherwise. The following are some examples.

**Invariants and covariants.** If $D_1 = \partial/\partial x_1$ and $D_2 = \partial/\partial x_2$, then

$$D_2^r D_1^{n-r} (\alpha_1 x_1 + \alpha_2 x_2)^n = n! \alpha_1^{n-r} \alpha_2^r,$$

where $\alpha_2^n$ is a special form of degree $n$, while the operation on the general form gives

$$D_2^r D_1^{n-r} f(x_1, x_2) = D_2^r D_1^{n-r} (a_0 x_1^n + na_1 x_1^{n-1} x_2 + \cdots ) = n! a_r.$$

We now transform our coordinates:

$$x_1 = \xi_1 X_1 + \eta_1 X_2,$$

$$x_2 = \xi_2 X_1 + \eta_2 X_2,$$

or

$$X_1 = \frac{1}{\Delta} (\eta_2 x_1 - \eta_1 x_2), \quad X_2 = \frac{1}{\Delta} (-\xi_2 x_1 + \xi_1 x_2),$$

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where
\[ \Delta = \begin{vmatrix} \xi_1 & \eta_1 \\
\xi_2 & \eta_2 \end{vmatrix} \neq 0. \]

Then if \( \partial / \partial X_1 = \overline{D}_1 \) and \( \partial / \partial X_2 = \overline{D}_2 \), we write
\[
\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial X_1} \frac{\partial X_1}{\partial x_1} + \frac{\partial f}{\partial X_2} \frac{\partial X_2}{\partial x_1}
\]
in symbolic form
\[
D_1f = \frac{1}{\Delta} \left( \overline{D}_1 \eta_2 - \overline{D}_2 \xi_2 \right) f,
\]
also
\[
D_2f = \frac{1}{\Delta} \left( - \overline{D}_1 \eta_1 + \overline{D}_2 \xi_2 \right) f;
\]
and if \( D' = \partial / \partial y \) we have
\[
(DD') = (D_1D'_2 - D_2D'_1)
\]
\[
= \frac{1}{\Delta^2} \left[ \left( \overline{D}_1 \eta_2 - \overline{D}_2 \xi_2 \right) \left( - \overline{D}_1' \eta_1 + \overline{D}_2' \xi_1 \right) - \left( - \overline{D}_1 \eta_1 + \overline{D}_2 \xi_1 \right) \left( \overline{D}_1' \eta_2 - \overline{D}_2' \xi_2 \right) \right]
\]
\[
= \frac{1}{\Delta^2} \begin{vmatrix} \eta_2 - \xi_2 & \eta_1 - \xi_1 \\
\overline{D}_1' & \overline{D}_1' \\
\overline{D}_2' & \overline{D}_2' \end{vmatrix} = \frac{1}{\Delta} \begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\
\overline{D}_2 & \overline{D}_2' \end{vmatrix}.
\]
For example
\[
f(x)_a = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4
\]
gives
\[
\frac{1}{2} \begin{vmatrix} D_1 & D_1' \\
D_2 & D_2' \end{vmatrix} \begin{vmatrix} f(x)_a f(y)_b \end{vmatrix}_{y-x} = (4!)^2(a_0a_4 - 4a_1a_3 + 3a_2^2),
\]
while
\[
\frac{1}{2} \begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\
\overline{D}_2 & \overline{D}_2' \end{vmatrix} \begin{vmatrix} F(X)_a F(Y)_b \end{vmatrix}_{y=X} = (4!)^2(A_0A_4 - 4A_1A_3 + 3A_2^2).
\]
Therefore
\[
A_0A_4 - 4A_1A_3 + 3A_2^2 = \Delta^4(a_0a_4 - 4a_1a_3 + 3a_2^2).\]
This is Cayley’s hyperdeterminant notation. This would be, in
the Aronhold symbolic notation,

\[
\frac{1}{2} \left| \begin{array}{cc}
D_1 & D_1' \\
\Delta_2 & \Delta_4
\end{array} \right| \alpha_4 \beta_4 \bigg|_{y=z} = \frac{1}{2} (4!)^2 (\alpha \beta)^4.
\]

Blissard’s umbral notation. Let \( D = d/dy \). Then the symbolic
form of MacLaurin’s theorem is

\[
F(x) = F(y) \left[ x + \frac{D^2}{1 \cdot 2} F(y) \right]_{y=0} x^2 + \cdots,
\]
or

\[
F(x) = \left( 1 + xD + \frac{x^2 D^2}{2!} + \cdots \right) F(y) \bigg|_{y=0}.
\]

Now if

\[
F(y) = 1 + B_1 y + \frac{B_3 y^3}{2!} + \frac{B_3 y^5}{3!} + \cdots, \quad (e^{B_y}),
\]
then

\[
F(y) \bigg|_{y=0} = 1; \quad DF(y) \bigg|_{y=0} = B_1; \quad D^2 F(y) \bigg|_{y=0} = B_2; \quad \cdots;
\]
and

\[
\left( 1 + nB_1 + \frac{n(n-1)}{1 \cdot 2} B_2 + \cdots + nB_{n-1} + B_n \right) - B_n = [1 + B]^n - B^n = 0
\]
becomes

\[
\left\{ \left( 1 + nD + \frac{n(n-1)}{2!} D^2 + \cdots \right) + nD^{n-1} + D^n \right\} F(y) \bigg|_{y=0} = \left\{ [1 + D]^n - D^n \right\} F(y) \bigg|_{y=0} = 0;
\]
and one of the principal identities used in Blissard’s theory be-
comes
\[ \{ [f(x + (D - 1)\theta) - f(x - \theta D)]F(y) \}_{y=0} = \theta \frac{df(x)}{dx}. \]

Blissard's remark, "An equation which has a representative quantity is not susceptible to any algebraic operation by which the indices would be affected," becomes

\[ (Df)^2 \neq D^2f. \]

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**ON FOURTH ORDER SELF-ADJOINT DIFFERENCE SYSTEMS**

**BY V. V. LATSHAW**

A linear difference expression for which the differential transform is self-adjoint (anti-self-adjoint) we shall call self-adjoint (anti-self-adjoint).† We choose two fourth order difference equations

\begin{align*}
L^+(u) &\equiv p(x)[u(x + 2) + u(x - 2)] \\
&\quad + \lambda[u(x + 1) + u(x - 1)] + R(x)u(x) = 0, \\
L^-(u) &\equiv p(x)[u(x + 2) - u(x - 2)] \\
&\quad + \lambda[u(x + 1) - u(x - 1)] = 0,
\end{align*}

where \( L^+(u) \) is self-adjoint and \( L^-(u) \) anti-self-adjoint for the range (\( x = a, a + 1, \cdots, b - 1; b - a \geq 4 \)). \( R(x) \) and \( p(x) \) are both real, \( p(x) \) being a non-vanishing periodic function of period two; \( \lambda \) is a parameter.

Let the functions \( (y_1, y_2, y_3, y_4) \) constitute a fundamental set of solutions for either (1) or (2), and \( (w_1, w_2, w_3, w_4) \) the set adjoint to it. The two sets are related by the equations

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