This system, together with the initial conditions, is satisfied by $P_i = 0, (i = 1, \ldots, k)$. Hence, on account of the uniqueness of the solution of (14) with given initial values, we conclude that $P_i = 0$, and the proof is complete.

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ON THE EXISTENCE OF LINEAR FUNCTIONALS DEFINED OVER LINEAR SPACES*

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1. Introduction. A function $q(x)$ with domain in a linear space $E$ and range in the set $R$ of real numbers is called a functional, and $q(x)$ is called linear, if

$$q(ax + by) = aq(x) + bq(y), \quad x, y \in E; a, b \in R. \quad (1)$$

We call a functional $r(x)$ an $r$-function (over $E$) if there exists a linear functional $f(x)$ with

$$f(x) \leq r(x), \quad x \in E. \quad (2)$$

Using a notation of Banach† we call a functional $p(x)$ a $p$-function if

$$p(tx) = tp(x), \quad t \geq 0, x \in E, \quad (3)$$

$$p(x + y) \leq p(x) + p(y), \quad x, y \in E. \quad (4)$$

A fundamental theorem of Banach (loc. cit., p. 29) can be stated as follows:

**Theorem** (Banach). Each $p$-function is an $r$-function.

In some problems‡ involving existence of linear functionals $f_i(x)$ having prescribed properties, there appears a functional $q(x)$ with the following significance: There exists a linear functional $f_i$ having the requisite properties if and only if there exists

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* Presented to the Society, September 8, 1937.
‡ The author intends to discuss these problems at some future time.
a linear functional \( f \) with \( f(x) \leq q(x) \), that is, if and only if \( q(x) \) is an \( r \)-function. If \( q(x) \) can be shown to be a \( p \)-function, the problem is solved by Banach’s theorem; if \( q(x) \) is not a \( p \)-function or one is unable to decide whether \( q(x) \) is a \( p \)-function, Banach’s theorem cannot be applied. These considerations, and the fact that it is easy to give examples of \( r \)-functions which are not \( q \)-functions, lead one to desire an analytic characterization of \( r \)-functions. In §2 we give such a characterization, and in §3 we give some closely related theorems.

2. Characterization of \( r \)-functions. We prove now the theorem:

**Theorem 1.** In order that a functional \( r(x) \) defined over \( E \) may be an \( r \)-function, it is necessary and sufficient that

\[
\text{g.l.b.} \sum_{n, t_k > 0; \Sigma x_k = 0} \frac{r(t_k x_k)}{t_k} \geq 0.
\]

In (5), \( \sum x_k \) stands for the sum \( x_1 + \cdots + x_n \) of elements \( x_k \in E \). To prove necessity, let \( r(x) \) be an \( r \)-function and let \( f(x) \) be a linear functional with \( f(x) \leq r(x) \) for all \( x \in E \). Then if \( n, t_1, t_2, \ldots, t_n > 0 \) and \( \sum x_k = 0 \), we have

\[
f(x_k) = f(t_k x_k)/t_k \leq r(t_k x_k)/t_k,
\]

so that

\[
0 = f(0) = f(\sum x_k) = \sum f(x_k) \leq \sum r(t_k x_k)/t_k,
\]

and (5) follows.

To prove sufficiency, let (5) hold and define the functional \( \phi^{(r)}(x) \) by the formula

\[
\phi^{(r)}(x) = \text{g.l.b.} \sum_{n, t_k > 0; \Sigma x_k = x} \frac{r(t_k x_k)}{t_k}, \quad x \in E.
\]

To show that \( \phi^{(r)}(x) \) exists (is finite) for each \( x \in E \), we observe that if \( n, t_1, \ldots, t_n > 0 \) and \( \sum x_k = x \), then \( x_1 + \cdots + x_n + (-x) = 0 \) and it follows from (5) that

\[
\sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} + \frac{r(-x)}{1} \geq 0,
\]

and hence
which implies that \(- r(-x) \leq \rho^{(r)}(x)\). If in the sum in the right member of (8) we put \(n = 1, t_1 = 1, x_1 = x\), we obtain \(\rho^{(r)}(x) \leq r(x)\). Therefore

\[
- r(-x) \leq \rho^{(r)}(x) \leq r(x), \quad x \in E.
\]

We prove next that \(\rho^{(r)}(x)\) is a \(P\)-function. If \(x \in E\) and \(t > 0\), then

\[
\rho^{(r)}(tx) = \text{g.l.b.} \left\{ \sum_{k=1}^{n} \frac{r(t_k x_k)}{t_k} \right\}
\]

\[
= t \text{ g.l.b.} \left\{ \sum_{k=1}^{n} \frac{r([t_k x_k/t_k])}{t_k} \right\}
\]

\[
= t \text{ g.l.b.} \left\{ \sum_{k=1}^{n} \frac{r(u_k y_k)}{u_k} \right\} = tp^{(r)}(x),
\]

so that \(\rho^{(r)}(tx) = tp^{(r)}(x)\) for \(t > 0\). Substitution of \(t = 2, x = 0\) in this formula gives \(\rho^{(r)}(0) = 0\). Therefore

\[
\rho^{(r)}(tx) = tp^{(r)}(x), \quad t \geq 0; x \in E.
\]

To prove that

\[
\rho^{(r)}(x + y) \leq \rho^{(r)}(x) + \rho^{(r)}(y), \quad x, y \in E,
\]

let \(x, y \in E\) be fixed and let \(\epsilon = 0\). Choose \(m, t_1, \ldots, t_m > 0\) and \(x_1, \ldots, x_m \in E\) such that \(\sum x_i = x\) and

\[
\sum_{j=1}^{m} r(t_j x_j)/t_j < \rho^{(r)}(x) + \epsilon;
\]

and choose \(n, u_1, \ldots, u_n > 0\) and \(y_1, \ldots, y_n \in E\) such that \(\sum y_k = y\) and

\[
\sum_{k=1}^{n} r(u_k y_k)/u_k < \rho^{(r)}(y) + \epsilon.
\]

Since \(m + n, t_j, u_k > 0\) and \(x_1 + \cdots + x_m + y_1 + \cdots + y_n = x + y\), it follows from the definition of \(\rho^{(r)}(x + y)\) that
The arbitrariness of \( \varepsilon > 0 \) gives (11). Thus \( \phi^{(r)}(x) \) is a \( \phi \)-function and it follows from Banach's theorem that there exists a linear functional \( f(x) \) with \( f(x) \leq \phi^{(r)}(x) \). Using (9), we obtain \( f(x) \leq r(x) \); thus \( r(x) \) is an \( r \)-function and Theorem 1 is proved.

3. Significance of \( \phi^{(r)}(x) \). From Theorem 1 and its proof, we obtain the first part of our next theorem.

**Theorem 2.** If \( r(x) \) is an \( r \)-function, then the functional \( \phi^{(r)}(x) \) defined by

\[
\phi^{(r)}(x) = \frac{\text{g.l.b.} \sum_{k=1}^{n} \frac{r(t_kx_k)}{t_k}}{x \in E},
\]

is a \( \phi \)-function with

\[
-r(x) \leq - \phi^{(r)}(x) \leq \phi^{(r)}(x) \leq r(x), \quad x \in E;
\]

moreover if \( \phi(x) \) is a \( \phi \)-function with \( \phi(x) \leq r(x) \) for all \( x \in E \), then

\[
- \phi^{(r)}(-x) \leq - \phi(-x) \leq \phi(x) \leq \phi^{(r)}(x), \quad x \in E.
\]

In establishing (13), we use (9) and the fact that, for any \( \phi \)-function, \( 0 = \phi(0) = \phi(x-x) \leq \phi(x) + \phi(-x) \) and hence \( - \phi(-x) \leq \phi(x) \) for all \( x \in E \). If \( \phi(x) \leq r(x) \); \( n, t_1, \cdots, t_n > 0 \); and \( \sum x_k = x \); then

\[
\phi(x) \leq \sum_{k=1}^{n} \phi(x_k) = \sum_{k=1}^{n} \frac{\phi(t_kx_k)}{t_k} \leq \sum_{k=1}^{n} \frac{r(t_kx_k)}{t_k}
\]

and \( \phi(x) \leq \phi^{(r)}(x) \) follows. The remaining inequalities in (14) follow easily, and Theorem 2 is proved. The gist of Theorem 2 is that \( \phi^{(r)}(x) \) is the "greatest" \( \phi \)-function \( \phi(x) \) with \( \phi(x) \leq r(x) \). In particular, if \( r(x) \) is a \( \phi \)-function, then \( \phi^{(r)}(x) = r(x) \).

Since each linear functional \( f(x) \) is a \( \phi \)-function, Theorem 2 implies the following theorem:

**Theorem 3.** If \( r(x) \) is an \( r \)-function and \( f(x) \) is a linear functional with \( f(x) \leq r(x) \), then

\[
-r(-x) \leq - \phi^{(r)}(-x) \leq f(x) \leq \phi^{(r)}(x) \leq r(x), \quad x \in E.
\]
It thus appears that the class of linear functionals $f(x)$ for which $f(x) \leq p^{(r)}(x)$ is identical with the class of linear functionals $f(x)$ for which $f(x) \leq r(x)$.

4. **Conclusion.** The functionals $q(x)$ mentioned in the introduction often have the property $q(tx) = tq(x)$ for $t \geq 0$, and $x \in E$. Hence it is of interest to note that if

$$r(tx) = tr(x), \quad t \geq 0, \quad x \in E,$$

then the criterion (5) that $r(x)$ be an $r$-function reduces to

$$\text{g.l.b.} \sum_{n > 0; \sum x_k = 0}^n r(x_k) \geq 0,$$

and that formula (12) for $p^{(r)}(x)$ reduces to

$$p^{(r)}(x) = \text{g.l.b.} \sum_{n > 0; \sum x_k = x}^n r(x_k), \quad x \in E.$$