ON CONTINUED FRACTIONS REPRESENTING CONSTANTS*

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1. Introduction. Let $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ be an infinite sequence of points $x = (x_1, x_2, x_3, \ldots, x_m)$ in a space $S$, and let $\phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_k(x)$ be single-valued real or complex functions over $S$. Then the functionally periodic continued fraction

$$1 + \frac{\phi_1(x^{(1)})}{1 + \frac{\phi_2(x^{(1)})}{1 + \cdots + 1 + \frac{\phi_k(x^{(1)})}{1 + \frac{\phi_1(x^{(2)})}{\phi_1(x^{(3)})}} + \cdots}}$$

is a function $f(\xi)$ of the sequence $\xi$. By a neighborhood of a sequence $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots$, we shall understand a set $N_\xi$ of sequences subject to the following conditions: (i) $\xi$ is in $N_\xi$; (ii) if $y^{(1)}, y^{(2)}, y^{(3)}, \ldots$ is in $N_\xi$, then $\eta_y: y^{(r+1)}, y^{(r+2)}, y^{(r+3)}, \ldots$ and $\xi_y: y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(r)}, x^{(r+1)}, x^{(r+2)}, x^{(r+3)}, \ldots$ are in $N_\xi$ for $r = 1, 2, 3, \ldots$.

Let $A_n(\xi)$ and $B_n(\xi)$ be the numerator and denominator, respectively, of the $n$th convergent of $f(\xi)$ as computed by means of the usual recursion formulas. Put

$$L(\xi, t) = B_{k-1}(\xi) t^2 + [\phi_k(x^{(1)}) B_{k-2}(\xi) - A_{k-1}(\xi)] t - \phi_k(x^{(1)}) A_{k-2}(\xi).$$

Then our principal theorem is as follows:

**Theorem 1.** Let there be a sequence $c: c^{(1)}, c^{(2)}, c^{(3)}, \ldots$, and a neighborhood $N_c$ of $c$, and a number $r$ having the following properties:

(a) $f(\xi)$ converges uniformly over $N_c$,
(b) $f(c) = r$,
(c) $L(\xi, r) = 0$ for every sequence $\xi$ in $N_c$,
(d) $\phi_i(x^{(v)}) \neq 0$, $(v = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots, k)$, for every sequence $\xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ in $N_c$.

When these conditions are fulfilled, $f(\xi) = r$ throughout $N_c$.

The proof of Theorem 1 is contained in §2; §3 contains a specialization and §4 an application of this theorem. In §5 continued fractions

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representing constants are obtained by means of certain transformations.*

2. **Proof of Theorem 1.** Let \( \eta : \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \ldots \) be any sequence in \( N_c \). Then \( \eta : y^{(r+1)}, y^{(r+2)}, y^{(r+3)}, \ldots \) is in \( N_c \), and \( f(\eta), (\nu = 0, 1, 2, \ldots; \eta_0 = \eta) \), converges by (a); and

\[
f(\eta) = \frac{A_{k-1}(\eta)f(\eta_{r+1}) + A_{k-2}(\eta_2)\phi_k(y^{(r+1)})}{B_{k-1}(\eta)f(\eta_{r+1}) + B_{k-2}(\eta_2)\phi_k(y^{(r+1)})},
\]

(1)

\[
f(\eta_{r+1}) = -\frac{B_{k-2}(\eta_2)f(\eta_r) - A_{k-2}(\eta_2)\phi_k(y^{(r+1)})}{B_{k-1}(\eta_r)f(\eta_r) - A_{k-1}(\eta_2)\phi_k(y^{(r+1)})}.
\]

The determinant of the matrix

\[
\begin{pmatrix}
A_{k-1}(\eta_2), & A_{k-2}(\eta_2)\phi_k(y^{(r+1)}) \\
B_{k-1}(\eta_2), & B_{k-2}(\eta_2)\phi_k(y^{(r+1)})
\end{pmatrix}
\]

is \( \pm \phi_1(y^{(r+1)})\phi_2(y^{(r+1)}) \cdots \phi_k(y^{(r+1)}) \) and is therefore \( \neq 0 \) by (d). Hence the denominators in (1) cannot vanish, for otherwise the numerators would also vanish, which is impossible. It then follows from (c) that if \( f(\eta_\nu) = r \) for one value of \( \nu \), then \( f(\eta_\nu) = r \) for all values of \( \nu (=0, 1, 2, 3, \ldots) \). In particular, if \( \xi_\nu \) is the sequence \( y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(\nu)}, c^{(r+1)}, c^{(r+2)}, c^{(r+3)}, \ldots \), then \( f(\xi_\nu) = r, (\nu = 1, 2, 3, \ldots) \).

Now by (a), for every \( \varepsilon > 0 \) there exists a \( K \) such that if \( n > K \), \( \rho = 1, 2, 3, \ldots \),

\[
| A_{n+\rho}(\xi_\nu)B_n(\xi_\nu) - A_n(\xi_\nu)B_{n+\rho}(\xi_\nu) | < \varepsilon
\]

for \( \nu = 1, 2, 3, \ldots \). Choose a fixed \( n > K \), and then choose \( \nu \) so large that \( A_n(\xi_\nu)/B_n(\xi_\nu) = A_n(\eta)/B_n(\eta) \). Then on allowing \( \rho \) to increase to \( \infty \) in (2) we find that

\[
| f(\xi_\nu) - A_n(\eta)/B_n(\eta) | \leq \varepsilon \quad \text{or} \quad | r - A_n(\eta)/B_n(\eta) | \leq \varepsilon
\]

if \( n > K \). That is, \( f(\eta) = r \). Since \( \eta \) was any sequence in \( N_c \) our theorem is proved.

3. Specialization of Theorem 1. Let the sequence \(c\) be such that \(f(c)\) is a periodic continued fraction of period \(k\). Let \(r, s\) be the roots of the quadratic equation \(L(c, t) = 0\). Then* in order for \(f(c)\) to converge to the value \(r\) the following two conditions are both necessary and sufficient, namely:

\[ B_{k-1}(c) \neq 0, \]

\[ r = s \text{ or else } \left| B_{k-1}(c)r + \phi_s(c^{(1)})B_{k-2}(c) \right| > \left| B_{k-1}(c)s + \phi_h(c^{(1)})B_{k-2}(c) \right| \text{ and } A_{\lambda}(c) - sB_{\lambda}(c) \neq 0, (\lambda = 0, 1, 2, \ldots, k-2). \]

An important and simple sufficient condition† for the uniform convergence of \(f(\xi)\) over \(N_e\) is that

\[ B(c(x^\nu)) \leq \frac{1}{4}, (i = 1, 2, 3, \ldots, k; \nu = 1, 2, 3, \ldots), \text{ for every sequence } \xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots \text{ in } N_e. \]

From these remarks and Theorem 1 we then have this result:

**Theorem 2.** Let there be a sequence \(c\) and a neighborhood \(N_e\) of \(c\) such that (\(\gamma\)) and conditions (c), (d) of Theorem 1 hold. Then if \(f(c)\) is a periodic continued fraction of period \(k\), we have \(f(\xi) = r\) throughout \(N_e\).

4. Application in the case where \(\phi_1, \phi_2, \phi_3, \ldots, \phi_k\) are polynomials.

If \(k = 1\), then \(L(\xi, t) = t^2 - t - \phi_1(x^{(1)})\), so that in order for (c) of Theorem 1 to hold \(\phi_1\) must be a constant, and \(f(\xi)\) reduces to an ordinary periodic continued fraction.

Let \(k = 2\). Then \(L(\xi, t) = t^2 + [\phi_2(x^{(1)}) - \phi_1(x^{(1)}) - 1]t - \phi_2(x^{(1)})\). We shall suppose that \(\phi_\nu(x) = \phi_\nu(x_1, x_2, x_3, \ldots, x_m), (\nu = 1, 2),\) are polynomials in the real or complex variables \(x_1, x_2, x_3, \ldots, x_m\). Let \(a, b\) be the constant terms, and \(G, H\) the coefficients of \(x_1^ax_2^b\cdots x_m^w\) in \(\phi_1\) and \(\phi_2\), respectively. Then (c) of Theorem 1 is equivalent to the relations

\[ (b-a)r - b = r(1-r), \quad (H-G)r - H = 0, \quad \text{all } G, H. \]

If \(r = 0\), then \(\phi_2 \equiv 0\), while if \(r = 1\), then \(\phi_1 \equiv 0\). Suppose \(r \neq 0, 1.\) Then if either \(G\) or \(H\) is 0, the other is 0 also, and if \(G = H\), their common value is 0. Hence (c) of Theorem 1 takes the form of the following identity:

\[ r\phi_1 \equiv (r - 1)(\phi_2 + r), \quad r \neq 0, 1. \]

On referring to Theorem 2 we now have this result:

**Theorem 3.** Let \(\phi_1(x)\) and \(\phi_2(x)\) be polynomials in the real or complex variables \(x_1, x_2, x_3, \ldots, x_m\) connected by the identity (3) with con-

† Perron, loc. cit., p. 262.
stant terms $a$ and $b$, respectively. Let $r$, in (3), and $s$ be the roots of the quadratic equation $t^2 + (b - a - 1)t - b = 0$ such that $r = s$ or else $|r + b| > |s + b|$, $s \neq 1$. Let $a$, $b$ be such that $|a| < \frac{1}{2}$, $|b| < \frac{1}{2}$, $a \neq 0$, $b \neq 0$. Then there exists a positive constant $R$ such that throughout the circle $|x_i^{(v)}| \leq R$, $(i = 1, 2, \cdots, m; v = 1, 2, \cdots)$, we have

\begin{equation}
1 + \frac{\phi_1(x^{(1)})}{1 + \frac{\phi_2(x^{(1)})}{1 + \frac{\phi_1(x^{(2)})}{1 + \frac{\phi_2(x^{(2)})}{1 + \cdots}}}} = r,
\end{equation}

\[x^{(r)} = (x_1^{(r)}, x_2^{(r)}, \cdots, x_m^{(r)}).
\]

In applying Theorem 2 we have taken $c^{(v)} = (0, 0, 0, \cdots, 0)$ in the sequence $c$. It is to be observed that, when this is done and Theorem 2 applies, the value of the continued fraction depends upon only the constant terms of the polynomials $\phi_1, \phi_2, \phi_3, \cdots, \phi_k$.

5. Singular continued fractions. Let $T$ be a transformation which carries the continued fraction $f = x_0 + K(x_i/1)$ into another continued fraction $Tf = x'_0 + K(x'_i/1)$ in such a way that when either $f$ or $Tf$ converges the other does also and their values are equal. We shall speak of such a transformation as a proper transformation of $f$. Suppose moreover that for some positive integer $n$ the elements $x_i$ of $f$ are subject to the condition

\begin{equation}
x_i = x'_i, \quad i = n, n+1, n+2, \cdots.
\end{equation}

This gives the following formal relation:

\[
x_0 + \frac{x_1}{1 + \cdots + \frac{x_{n-1}}{g_n}} = x'_0 + \frac{x'_1}{1 + \cdots + \frac{x'_{n-1}}{g_n}},
\]

from which one may compute the value of the continued fraction

\[g_n = 1 + \frac{x_n}{1 + \frac{x_{n+1}}{1 + \cdots}}
\]

when the latter converges.

The procedure outlined above will now be carried out for the following proper transformation:* 

\[x'_0 = x_0 + x_1, \quad x'_1 = -x_1, \quad x'_2 = (1 + x_3)/x_2;
\]

\[T_2: \quad x'_{2n+1} = x_{2n+1}, \quad x'_{2n+2} = (1 + x_{2n+1})(1 + x_{2n+3})/x_{2n+2}, \quad n = 1, 2, 3, \cdots; \quad x_n \neq 0, -1 \text{ if } n > 0.
\]

In this case the relations (5) are satisfied if and only if

\* Leighton and Wall, loc. cit., p. 277.
where if \( n = 0 \) the first of these relations is to be replaced by \( x_2^2 = (1 + x_0) \). When \( n = 0 \) we have the relation

\[ x_0 + x_1/g_2 = x_0 + x_1 - x_1/g_2 \]

from which to compute \( g_2 \). It follows that, if \( f \) converges, \( g_2 \) must converge and have the value 2; and if \( g_2 \) converges to a value different from 0, \( f \) must converge and \( g_2 = 2 \). Moreover, it is impossible for \( g_2 \) to have the value \( \infty \), for that would imply that \( f = x_0 \) while \( Tf = x_0 + x_1 \neq f \). If we now write out the continued fraction \( g_2 \) and make a change in notation, the following theorem results.

**Theorem 4.** If \( x_1, x_2, x_3, \ldots \) are arbitrary complex numbers \( \neq 0, -1 \), then the continued fraction

\[
1 + \frac{e_1(1 + x_1)^{1/2}}{1 + \frac{x_1}{1 + \frac{e_2[(1 + x_1)(1 + x_2)]^{1/2}}{1 + \frac{x_2}{1 + \frac{e_3[(1 + x_2)(1 + x_3)]^{1/2}}{1 + \frac{x_3}{1 + \cdots}}}}},
\]

(7)

has one of the values 0 or 2 whenever it converges, and it cannot diverge to \( \infty \).

It is interesting to observe that if \( e_i = +1 \), (7) is the formal expansion of 2 into a continued fraction by means of the identity

\[
1 + \frac{(1 + t)^{1/2}}{t} = \frac{1}{1 + (1 + t)^{1/2}}.
\]

As a special case we have the expansion

\[
(1 + N)^{1/2} = 1 + \frac{N + 1}{1 + \frac{N}{1 + \frac{N + 1}{1 + \cdots}}},
\]

which is valid if \( N \) is a positive integer.

The transformation \( T_2 \) is one of an infinite group of transformations discussed by the writer* elsewhere in this Bulletin. If one obtains the singular continued fractions corresponding to the case \( m = 3 \) (in the notation of §3, p. 589, of that article), the following three theorems result.

* Wall, loc. cit.
Theorem 5. If the continued fraction

\[ 1 - \frac{x_1}{1 - \frac{x_2}{1 - \frac{x_3}{1 - \ldots}}} = \frac{(x_1^2 - x_1 + 1)}{x_1} \frac{x_2}{1 - \frac{(x_2^2 - x_2 + 1)}{1}} \]

converges, its value is \((1 \pm 3^{1/2})/2\).

Theorem 6. If the continued fraction

\[ 1 - \frac{e_1}{1 - \frac{x_1}{1 - \frac{(2 - x_1)}{1 - \frac{e_2}{1 - \frac{x_2}{1 - \ldots}}}}} = \frac{2 - x_1}{x_1} \frac{e_2}{1 - \frac{2 - x_2}{1}} \frac{e_3}{1 - \ldots} \]

converges, its value is 0 or 1.

Theorem 7. If the continued fraction

\[ 1 - \frac{x_1}{1 - \frac{(1 - 2x_1)}{1 - \frac{x_2}{1 - \frac{(1 - 2x_2)}{1 - \ldots}}} = 1} \frac{x_2}{1 - \frac{1}{1 - \ldots}} \]

converges, its value is 0 or \(1/2\).

The proofs of these theorems are along the lines of the proof of Theorem 4, and will be omitted.