A REMARK ON REPRESENTATIONS OF GROUPS*

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The purpose of this short note is to remark that we can state an analog of a famous theorem of Frobenius† on the induced characters of a finite group also for the representations of a general group.‡ This extension has not yet been explicitly stated, so far as I know, although it can be quite easily verified.

Let $g$ be a group, and let $h$ be a subgroup (of a finite or infinite index) of $g$.

**DEFINITION.** Let $F(x)$ and $f(\xi)$ be almost periodic (a. p.) functions (with complex numbers as values) on $g$ and $h$ respectively. Then we define the compositions of $F(x)$ and $f(\xi)$ by

$$f \times F(x) = M_{h} [f(\xi)F(\xi^{-1}x)],$$

$$F \times f(x) = M_{h} [F(x\xi^{-1})f(\xi)];$$

where $M_{h}$ means the construction of the mean with respect to a variable $\xi$ in $h$. Here $f \times F(x)$ and $F \times f(x)$ are a. p. functions on $g$, and they are linear with respect to both factors, $f(\xi)$ and $F(x)$.

If $h_1, h_2, h_3$ are three subgroups of $g$ such that $h_i \subseteq h_k$ or $h_i \supseteq h_k$ for every $i, k = 1, 2, 3$, then

$$(f_1 \times f_2) \times f_3 = f_1 \times (f_2 \times f_3)$$

for a. p. functions $f_1(\xi_1), f_2(\xi_2), f_3(\xi_3)$ on $h_1, h_2, h_3$ respectively.

(Both sides of the equality are a. p. functions on the greatest among the $h_i$.) This product we denote by $f_1 \times f_2 \times f_3$.

All these statements we can prove by a procedure similar to that

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By a representation of a group we understand always a bounded one in the field of complex numbers.
of von Neumann in the paper cited. Therefore we can consider the ring $R_b$ of a. p. functions on $b$ as a (right and left) operator-ring of the ring $R_a$ of a. p. functions on $a$. If $M$ and $m$ are submoduli of $R_a$ and $R_b$ respectively, then we denote by $M \times m$ the submodule of $R_a$ generated by the elements $(F \times f(x), F(x) \epsilon M, f(x) \epsilon m)$. We define $m \times M$ in a similar manner.

**Definition.** Let $n$ be a left ideal of $R_b$ with a finite rank with respect to the field $\Omega$ of complex numbers.* Then $R_b \times n$ is obviously a left ideal of $R_a$ (with a finite or infinite rank with respect to $\Omega$). We call $R_b \times n$ the left ideal of $R_a$ induced by $n$.

As is well known, there is an idempotent element $c(\xi)$ in $R_b$ such that $n = R_b \times c$; it is $f \times c = f$ for every $f(x)$ in $n$. If $F(x) \epsilon R_b$ and $f(\xi) \epsilon n$, then we have $F \times f = F \times (f \times c) = (F \times f) \times c \epsilon R_b \times c$. Therefore $R_b \times n \subseteq R_a \times c$, and this implies $R_b \times n = R_b \times c$.

The ideal $R_b \times n$ consists of all functions $G(x) \epsilon R_b$ such that for every $x \epsilon a$ the function $G(x) \epsilon a \times n$ lies in $n$.

Let $G(x)$ have the property stated above. Then

$$M_{\eta \gamma} [G(x\eta^{-1})c(\eta)] = G(x\xi);$$

in particular,

$$G \times c(x) = M_{\eta \gamma} [G(x\eta^{-1})c(\eta)] = G(x),$$

that is, $G(x) \epsilon R_b \times c = R_b \times n$.

The other half of the statement is obvious.

Now we have the following theorem:

**Theorem.** Let $n$ be a minimal left ideal of $R_b$, and $b$ the irreducible representation of $b$ defined by $n$. Let $D$ be an irreducible representation of $a$, and $\equiv$ the two-sided ideal of $R_a$ belonging to $D$.† We denote by $D(b)$ a representation of $b$ formed by the matrices in $D$ which correspond to the elements of $b$. If the number of the irreducible constituents of $D(b)$ equivalent to $b$ is $g$, then the representation of $a$ defined by the $\equiv$-component $\equiv \times n = \equiv \times n$ of the induced left ideal $\equiv = R_b \times n$ consists of just $g$ irreducible constituents (equivalent to $D$).

**Proof.**‡ Let $E(x)$ denote the principal unit of $\equiv$, and put

* Note that a submodule of $R_b$ ($R_a$) with a finite rank with respect to $\Omega$ is a left ideal if and only if it is an $b$-($a$-)left-module; where we define the multiplication of $\alpha \epsilon b$ and $f(\xi) \epsilon R_b$, $(\alpha \epsilon b$ and $F(x) \epsilon R_a)$, by $\alpha \cdot f(\xi) = f(\alpha^{-1} \xi)$, $(a \cdot F(x) = F(a^{-1} x))$. Then we have $(\alpha \cdot f) \times F = \alpha \cdot (F \times f)$, $(a \cdot F) \times f = a \cdot (F \times f)$, and so on.

† Linear aggregates of the matric elements of $D$ form a two sided ideal of $R_a$, which is isomorphic to a matric ring.

‡ The following proof is only a slight modification of the proof in Weyl, loc. cit.
If we denote the degree of the representation $\mathfrak{b}$ by $r$, then the two-sided ideal $\mathfrak{s}$ of $\mathfrak{R}_b$ belonging to $\mathfrak{b}$ is a direct sum of $r$ minimal left ideals operator-isomorphic to $\mathfrak{n}$:

$$\mathfrak{s} = \mathfrak{n}^{(1)} + \mathfrak{n}^{(2)} + \cdots + \mathfrak{n}^{(r)}, \quad \mathfrak{n}^{(i)} \cong \mathfrak{n}.$$ 

Put $\mathfrak{R}^{(i)} = \mathfrak{R}_b \times \mathfrak{n}^{(i)}$ and $\mathfrak{H}^{(i)} = \mathfrak{S} \times \mathfrak{R}^{(i)} = \mathfrak{S} \times \mathfrak{n}^{(i)}$. It is easy to see that $\mathfrak{R}_b \times \mathfrak{s}$ is the direct sum $\mathfrak{R}^{(1)} + \mathfrak{R}^{(2)} + \cdots + \mathfrak{R}^{(r)}$, and also that $\mathfrak{S} \times \mathfrak{R}_b \times \mathfrak{s} = \mathfrak{S} \times \mathfrak{s} = \mathfrak{H}^{(1)} + \mathfrak{H}^{(2)} + \cdots + \mathfrak{H}^{(r)}$. Now suppose that $\mathfrak{H}$ is a direct sum of $h$ minimal left ideals. (Our purpose is to show $h = g$.) Then each of $\mathfrak{H}^{(i)}$ has the same property: $\mathfrak{H}^{(i)} = \mathfrak{D}_1^{(i)} + \mathfrak{D}_2^{(i)} + \cdots + \mathfrak{D}_h^{(i)}$, for it is operator-isomorphic to $\mathfrak{H}$.

Let $e(\xi)$ be the principal unit of $\mathfrak{s}$. We have $\mathfrak{S} \times \mathfrak{s} = \mathfrak{R}_b \times (e \times E)$ and $e \times E = E \times e \times E = E \times e$, \quad $(e \times E)^2 = E \times e \times E = e \times E$.

Moreover

$$\mathfrak{S} = \mathfrak{S} \times \mathfrak{s} + \mathfrak{S}^* = \mathfrak{S}_1^{(1)} + \mathfrak{S}_2^{(1)} + \cdots + \mathfrak{S}_h^{(r)} + \mathfrak{S}^*$$

for a suitably chosen left ideal $\mathfrak{S}^*$ of $\mathfrak{S}$. From this decomposition we see in the usual manner that $(e \times E) \times \mathfrak{R}_b \times (e \times E)$ is a matric ring of degree $rh$ (over $\Omega$):

$$(e \times E) \times \mathfrak{R}_b \times (e \times E) = \sum_{h, i=1, \ldots, r, \mu, \lambda=1, \ldots, h} C_{\mu \lambda}^{(k)(i)} \Omega,$$

$$(e \times E) \times \mathfrak{S}^{(i)} = \sum_{h, \mu} C_{\mu \lambda}^{(k)(i)} \Omega,$$

($C_{\mu \lambda}^{(k)(i)}(r)$ being matric units). Here $(e \times E) \times \mathfrak{S}^{(i)} = e \times (E \times \mathfrak{S}^{(i)}) = e \times \mathfrak{S}^{(i)}$, and therefore $(e \times \mathfrak{S}^{(i)}; \Omega) = rh$.

On the other hand $\mathfrak{S}^{(i)}$ is, considered as an $\mathfrak{h}$-left-module, completely reducible, for it defines a representation of $\mathfrak{h}$ equivalent to $\mathfrak{D}(\mathfrak{h})$. Let $\mathfrak{S}^{(i)} = \mathfrak{M}_1 + \mathfrak{M}_2 + \cdots + \mathfrak{M}_t$ be its decomposition into simple (minimal) submoduli. According to our assumption just $g$ of $\mathfrak{M}_t$ are operator-isomorphic to $\mathfrak{n}$ with respect to $\mathfrak{n}$, and therefore also with respect to $\mathfrak{R}_b$. \footnote{A submodule, with a finite rank, of $\mathfrak{R}_b$ is an $\mathfrak{h}$-left-module if and only if it is an $\mathfrak{h}$-left-module. Two such moduli are operator-isomorphic with respect to $\mathfrak{R}_b$ if and only if they are so with respect to $\mathfrak{n}$. The same holds for the isomorphism between such a module and a left ideal of $\mathfrak{R}_b$.} This implies $(e \times \mathfrak{S}^{(i)}; \Omega) = rg$.

Comparing with the above result, we obtain $h = g$. 

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