MATRIC CONJUGATES IN A RING $R(A)$*

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1. **Introduction.** The concept of conjugate sets of matrices has undergone several modifications since first defined by Taber. Given a matrix M_0, Taber† defined a set $M_1, M_2, \cdots, M_{n-1}$ to be conjugate to M_0 if (a) the M_i are commutative, (b) they have a common characteristic equation, (c) their elementary symmetric functions are scalars and equal to the elementary symmetric functions of the roots of their characteristic equation.

In Taber's paper, the latent roots of M_0 were assumed to be distinct. Franklin‡ generalized the definition so that this restriction is unnecessary. A set $M_1, M_2, \cdots, M_{n-1}$ is conjugate in the sense of Franklin if (a) the M_i are commutative, (b) their elementary symmetric functions are the elementary symmetric functions of the latent roots of M_0.

Further extension of the concept was made by Sokolnikoff.§ Given a matrix M_0 whose minimum equation is $g(x)=\prod_{i=1}^{\nu}(x-\rho_i)^{\pi_i}=0$, $(\sum\pi_i=m)$, a set $M_1, M_2, \cdots, M_{n-1}$ is conjugate to M_0 with respect to $g(x)=0$, if (a) each M_i is expressible as a polynomial in M_0 with coefficients in the field formed by adjoining the roots ρ_i and the π_ith roots of unity to the field of the elements of M_0, (b) the elementary symmetric functions of the M_i are the elementary symmetric functions of the roots of $g(x)=0$.

Hermann‖ has used the term conjugate in an even broader sense to denote a set of matrices M_i whose elementary symmetric functions are scalars. That is, the M_i are conjugate with respect to any given polynomial $F(x)$ in that their elementary symmetric functions are the elementary symmetric functions of the roots of $F(x)=0$.

In this paper we propose, by use of the principal idempotent elements of a matrix A, to obtain conjugates of each of the above types corresponding to a restricted class of matrices, namely, any given matrix in the ring $R(A)$, where A is any matrix with simple latent roots. The symbolism employed enables us to write down immedi-

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† H. Taber, American Journal of Mathematics, vol. 13 (1891), pp. 157–172.

‡ P. Franklin, Annals of Mathematics, (2), vol. 23 (1921), pp. 97–100.

§ E. S. Sokolnikoff, American Journal of Mathematics, vol. 35 (1933), pp. 167–180.

‖ A. Hermann, Compositio Mathematica, vol. 1 (1934), pp. 284–302.

ately a complete set of conjugates of any of these types, thus avoiding much of the laborious effort involved in former methods. In §6, a set is obtained conjugate with respect to the minimum equation of M_0, but more general than the Sokolnikoff type in that the M_i are not expressible as polynomials in M_0.*

2. **Principal idempotent elements.** Let the minimum equation of a matrix A of order n be

$$\psi(\lambda) \equiv \prod_{i=1}^{\mu} (\lambda - \alpha_i)^{\nu_i} = 0, \qquad \sum \nu_i = \nu,$$

where the α_i are distinct. If $\psi(\lambda) \equiv (\lambda - \alpha_i)^{\nu_i}\pi_i(\lambda)$, then the μ functions $\pi_i(\lambda)$ are relatively prime to each other, and there exist μ functions $\rho_i(\lambda)$ such that $\sum_{i=1}^{\mu}\rho_i(\lambda)\pi_i(\lambda) \equiv 1$. The μ functions $e_i(A) = \rho_i(A)\pi_i(A)$, hereafter denoted merely by e_i, are known as the *principal idempotent elements* of A corresponding to α_i. They are linearly independent, each is different from zero, and they satisfy the relations

$$\sum e_i = 1, \qquad e_i^2 = e_i, \qquad e_i e_j = 0, \qquad i \neq j.$$

In this paper we shall assume that A has simple latent roots only, so that $\mu = n$, and the ρ_i are constants. We shall use the symbol $(k_1, k_2, \cdots, k_n)$ to denote the matrix $\sum_{i=1}^{n} k_i e_i$.

3. **Conjugates in the sense of Taber and of Franklin.** It is well known† that the characteristic equation of the matrix $(k_1, k_2, \cdots, k_n)$ is $\prod_{i=1}^{n}(\lambda - k_i) = 0$. If, then, we cyclicly permute the β_i in the matrix $M_0 = (\beta_1, \beta_2, \cdots, \beta_n)$, (where the β_i are not necessarily distinct), we obtain a set

$$M_i = (\beta_{i+1}, \beta_{i+2}, \cdots, \beta_i), \qquad i = 1, 2, \cdots, n-1,$$

having a common characteristic equation $\prod_{j=1}^{n}(\lambda - \beta_j) = 0$. Since the M_i also satisfy the other conditions imposed by Taber, they constitute a conjugate set in the sense of Taber. The set is more general than that exhibited by Taber in that the latent roots of M_0 are not necessarily distinct.

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* In all of these types it is implied that the matrices M_i are all of a specified order n. A. R. Richardson, in a recent paper (Quarterly Journal of Mathematics, Oxford Series, vol. 7 (1936), pp. 256–270), has exhibited a set of n matrices each of order $n!$ which (a) satisfy the same minimum equation $f_n(x) = 0$, (b) are commutative, and (c) whose elementary symmetric functions are equal to the elementary symmetric functions of the roots of $f_n(x) = 0$.

† Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, New York, 1934, p. 26.

To obtain a set conjugate in the sense of Franklin, we write the symbols for $M_0, M_1, M_2, \cdots, M_{n-1}$ in a square array so that each β_j, $(j=1, 2, \cdots, n)$, occurs once and only once in each column of the array. This insures that the elementary symmetric functions of the M_i shall be the elementary symmetric functions of the β_j, the latter being the roots of the characteristic equation of M_0.

4. **Hermann's conjugates.** Let $F(x)$ be any polynomial with scalar coefficients, having distinct zeros $\beta_1, \beta_2, \cdots, \beta_t$. If we choose any n of these, $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n}$ (not necessarily distinct), the matrix $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ is a root of $F(x)=0$; the matrix $M_i=(\beta_{i_1+i}, \beta_{i_2+i}, \cdots, \beta_{i_n+i})$, i_k+i being reduced modulo t, is also a root of $F(x)=0$, and is called by Hermann the ith *conjugate* of M_0. The matrices M_i are commutative, and their elementary symmetric functions are equal to those of the roots of $F(x)=0$.

5. **Conjugates in the sense of Sokolnikoff.** In order to obtain conjugates of this type, two theorems are necessary.

THEOREM 1. *If among the β's in $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ there are m which are distinct, say $\beta_1, \beta_2, \cdots, \beta_m$, then the minimum equation of M_0 is $\prod_{j=1}^{m}(\lambda-\beta_j)=0$.**

If we now let the polynomial $F(x)$ of the preceding section be $\prod_{j=1}^{m}(\lambda-\beta_j)$ and form the conjugates M_i, $(i=1, 2, \cdots, m-1)$, according to the rule indicated in that section, we can prove the following theorem:

THEOREM 2. *Each M_i can be expressed as a polynomial in M_0, with coefficients in the field formed by adjoining the β_i to the field of the elements of M_0.*

We have $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ and $M_i=(\beta_{i_1+i}, \beta_{i_2+i}, \cdots, \beta_{i_n+i})$, $(i=1, 2, \cdots, m-1)$. Assume that each M_i can be expressed in the form $M_i=\sum_{k=1}^{m} a_k M_0^{m-k}$. Since the e_i are linearly independent, in determining the a_k we are led to a system of m non-homogeneous equations whose determinant is

$$\pm\Delta = \begin{vmatrix} \beta_1^{m-1} & \beta_1^{m-2} & \cdots & 1 \\ \beta_2^{m-1} & \beta_2^{m-2} & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdot \\ \beta_m^{m-1} & \beta_m^{m-2} & \cdots & 1 \end{vmatrix} = \prod_{i<j}(\beta_i-\beta_j).$$

* Wedderburn, loc. cit

The β_j being distinct, Δ is not zero, and the a_k may be uniquely determined.

Thus the set $M_0, M_1, \cdots, M_{m-1}$ may be considered a very elementary set in the sense of Sokolnikoff. In fact, this is the special case referred to by Sokolnikoff in which the roots of the minimum equation are distinct. It can be readily shown that if $M_1=\Theta(M_0)$, then $M_i=\Theta^{i-1}(M_0)$, $(i=1, 2, \cdots, m-1)$, and $M_0=\Theta^m(M_0)$.*

6. **Other conjugate sets.** If we merely permute the β's in $M_0=(\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_n})$ to form the m matrices $M_0, M_1, \cdots, M_{m-1}$, in such a way that distinct β's shall occupy the jth place in each M_i, the set obtained is conjugate relative to the minimum equation of M_0, but, in general, it is impossible to express the other matrices as polynomials in M_0. For example, if $M_0=(\beta_1, \beta_1, \beta_2, \beta_3, \beta_1)$ we may choose $M_1=(\beta_3, \beta_3, \beta_3, \beta_2, \beta_3)$ and $M_2=(\beta_2, \beta_2, \beta_1, \beta_1, \beta_2)$. The set is conjugate relative to $\prod_{j=1}^{3}(\lambda-\beta_j)=0$, but any attempt to express M_1 or M_2 as polynomials in M_0 leads to a system of inconsistent equations. It will be observed that these matrices do not, in general, have a common minimum equation, a property which was possessed by the set obtained in the preceding section.

Finally, we note that this set and that of the Franklin type are not uniquely determined. In the other types described, the law by which M_i was obtained determines the latter uniquely.†

UNIVERSITY OF ALABAMA

* Compare Sokolnikoff, loc. cit., corollary, p. 175.

† This is not meant to imply that only one set exists corresponding to M_0, but rather that the law, *as stated*, yields a unique set. Other sets may be determined by a modification of the law. For example, if $M_0=(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3)$, we have the sets

$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_1, \beta_3, \beta_3, \beta_4), (\beta_3, \beta_2, \beta_4, \beta_4, \beta_1), (\beta_4, \beta_3, \beta_1, \beta_1, \beta_2)\};$$
$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_3, \beta_4, \beta_4, \beta_1), (\beta_3, \beta_2, \beta_1, \beta_1, \beta_4), (\beta_4, \beta_1, \beta_3, \beta_3, \beta_2)\};$$
$$\{(\beta_1, \beta_4, \beta_2, \beta_2, \beta_3), (\beta_2, \beta_3, \beta_1, \beta_1, \beta_4), (\beta_3, \beta_1, \beta_4, \beta_4, \beta_2), (\beta_4, \beta_2, \beta_3, \beta_3, \beta_1)\},$$

all three of which satisfy the requirements of a conjugate set in the sense of Sokolnikoff.