and as above this is sufficient that postulates I–V be satisfied.

With this definition of \( A : B, A \supset B \) becomes the usual inclusion relation of the algebra of classes [5].

REFERENCES


CALIFORNIA INSTITUTE OF TECHNOLOGY

A NOTE ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS

FRITZ JOHN

Let \( u(x_1, \ldots, x_n) \) denote a twice continuously differentiable function of \( x_1, \ldots, x_n \) in some region \( R \). We write \( \partial u/\partial x_i = u_i, \partial^2 u/\partial x_i \partial x_k = u_{ik} \), and occasionally \( (x) \) for \( (x_1, \ldots, x_n) \). A point \( (c) = (c_1, \ldots, c_n) \) of \( R \) may be called a proper maximum of \( u \), if

\[
\begin{align*}
    u_i(c) &= 0 \quad \text{for} \quad i = 1, \ldots, n, \\
    \sum_{i,k} u_{ik}(c) \xi_i \xi_k &< 0 \quad \text{for all} \quad (\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0).
\end{align*}
\]

A partial differential equation

\[
\sum_{i,k} a_{ik}(x) u_{ik}(x) + \sum_i b_i(x) u_i(x) = 0
\]

(where the \( a_{ik} \) and \( b_i \) are defined in \( R \)) is called elliptic if for every \( (x) \) of \( R \)

\[
\sum_{i,k} a_{ik}(x) \xi_i \xi_k \geq 0
\]

for all \( (\xi_1, \ldots, \xi_n) \) and \( < 0 \) for some \( (\xi_1, \ldots, \xi_n) \).
It is well known, that a solution \( u \) of (1) can not have a proper maximum.\(^*\) For if \( u \) had a proper maximum at \( (c_1, \ldots, c_n) \), then \( \sum_{i,k} a_{ik}(c)u_{ik}(c) = 0. \) If \( A \) and \( U \) denote respectively the matrices \( (a_{ik}(c)) \) and \( (u_{ik}(c)) \), this may be written: Trace \( (A \cdot U) = 0. \) By a suitable orthogonal transformation \( A \) may be transformed into a diagonal matrix \( A' = (a'_i \delta_{ik}) \), \( U \) going over into \( U' = (u'_i) \) by the same transformation. As the trace of \( A \cdot U \) is preserved, we have \( \sum a'_i u'_{ik} = 0; \) on the other hand, as \( A' \) still belongs to a semi-definite quadratic form and \( U' \) to a negative definite one, we have \( a'_i \geq 0 \) for \( i = 1, \ldots, n, \) but \( < 0 \) for some \( i, \) and \( u'_{ik} < 0 \) for all \( i. \) This leads to a contradiction.

A second important property of the solutions \( u \) of (1) is that they form a module, that is, that every linear combination with constant coefficients is again a solution.

We shall prove that these two properties are also sufficient to characterize a family of functions as solutions of an elliptic equation (1).

**Theorem.** Let \( F \) be any family of twice continuously differentiable functions \( u(x_1, \ldots, x_n) \) defined in \( \mathbb{R}, \) such that

(a) the functions of \( F \) form a module,

(b) no function of \( F \) has a proper maximum.

Then there is an elliptic differential equation (1) satisfied by all functions of \( F. \)

**Proof.** Let \( (c) = (c_1, \ldots, c_n) \) be a point of \( \mathbb{R}. \) Let \( \phi \) be the submodule of functions \( u \) of \( F \) for which \( u_{ik}(c) = 0 \) for \( i = 1, \ldots, n. \) Let \( Q(\xi_1, \ldots, \xi_n) \) denote the quadratic form \( \sum_{i,k} u_{ik}(c)\xi_i \xi_k \) for any \( u \) in \( \phi; \) \( Q \) is certainly not negative definite. These quadratic forms form again a module \( M. \) Let \( Q_1, Q_2, \ldots, Q_m \) form a basis of this module \( (m \leq n(n+1)/2), \) such that for every \( Q \) of \( M \)

\[
Q(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \ldots, \xi_n)
\]

with certain constants \( \lambda_i. \) We know that for every \( (\lambda_1, \ldots, \lambda_m) \) there are \( (\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0) \) such that

\[
\sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \ldots, \xi_n) \geq 0.
\]

From this we can easily conclude that the \( Q_i \) satisfy a linear relation

with positive coefficients.* For let $\Sigma$ denote the set of points with coordinates $(Q_1(\xi), Q_2(\xi), \ldots, Q_m(\xi))$ in an $m$-dimensional space, where $(\xi) = (\xi_1, \ldots, \xi_n)$ varies over all points of the unit sphere $\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = 1$. The set $\Sigma$ is closed and finite. The relation (2) may be interpreted as stating that every half-space bounded by a plane through the origin contains points of $\Sigma$. Thus the origin is contained in the convex extension of $\Sigma$. Then there exists a finite set of points $P_1, \ldots, P_r$ of $\Sigma$ and positive numbers $\mu_1, \ldots, \mu_r$, such that the origin is the center of mass of the masses $\mu_i$ placed at the vertices $P_i$.† Let $(\xi_1^i, \ldots, \xi_n^i)$ be the point $(\xi_1, \ldots, \xi_n)$ corresponding to $P_i$. Then

$$\sum_{j=1}^r \mu_j Q_k(\xi_1^j, \ldots, \xi_n^j) = 0$$

for $k = 1, \ldots, m$, and consequently

$$\sum_{j=1}^r \mu_j Q(\xi_1^j, \ldots, \xi_n^j) = 0$$

for every $Q$ in $M$. Thus

$$\sum_{j=1}^r \sum_{i,k} \mu_j a_{ik}(c_1, \ldots, c_n) \xi_1^j \xi_k^j = 0$$

for $u$ in $\phi$. Putting $\sum_{j=1}^r \mu_j a_{ik}(c_1, \ldots, c_n) \xi_1^j \xi_k^j = a_{ik}(c)$, we have

$$\sum_{i,k} a_{ik}(c) u_{ik}(c) = 0$$

for $u$ in $\phi$. Besides

$$\sum_{i,k} a_{ik}(c) \xi_1^i \xi_k = \sum_{i,j} \mu_j (\xi_1^i \xi_j)^2 \geq 0$$

and $> 0$ for some $(\xi_1, \ldots, \xi_n)$.

Now let $u(x_1, \ldots, x_n)$ denote an arbitrary function of $F$. The vectors $(y_1, \ldots, y_n) = (u_1(c), u_2(c), \ldots, u_n(c))$ again form a module $N$, if $u$ varies over $F$ for fixed $(c)$. Let $(y_1^1, y_1^2, \ldots, y_1^s), (y_2^1, \ldots, y_2^s), \ldots, (y_n^1, \ldots, y_n^s)$ form a basis of $N, (s \leq n)$. Without restriction of generality we may assume that this basis forms an orthogonal system:

$$\sum_{l=1}^n y_i^l y_k^l = \delta_{ik}, \quad i, k = 1, \ldots, s.$$  

---

Let \( u^1(x, c), u^2(x, c), \ldots, u^s(x, c) \) be the functions of \( F \) corresponding to the vectors

\[
u^k(c, c) = y^k.
\]

Then for every \( u \) in \( F \)

\[
u_i(c) = \sum_{j=1}^{s} \lambda_j y^j_i
\]

with

\[
\lambda_j = \sum_{k=1}^{n} u_k(c) y^j_k.
\]

Thus

\[
u_i(c) = \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u^j_k(c, c) u^i_j(c, c).
\]

Consider the function

\[
\bar{u}(x) = u(x) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u^j_k(c, c) u^i_j(c, c).
\]

Then \( \bar{u} \) is in \( F \). We have

\[
\bar{u}_i(c) = u_i(c) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u^j_k(c, c) u^i_j(c, c) = 0.
\]

Hence \( \bar{u} \) is in \( \phi \). Consequently

\[
0 = \sum_{i, k} a_{ik}(c) \bar{u}_{ik}(c) = \sum_{i, h} a_{ih}(c) u_{ih}(c) - \sum_{i, j, h} a_{ih}(c) u_k(c) u^j_k(c, c) u^i_j(c, c).
\]

Thus \( u(x) \) satisfies the elliptic equation

\[
0 = \sum_{i, h} a_{ih}(x) u_{ih}(x) - \sum_{i, j, h} a_{ih}(x) u^j_k(x, x) u^i_j(x, x) u_k(x).
\]

University of Kentucky