SOME THEOREMS ON SUBSEQUENCES†

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It is obvious that, for any real sequence for which the sum \( \Sigma \) of the moduli of its elements exists and is finite, there exists a subsequence such that the modulus of the sum of its elements is not less than \( \Sigma/2 \). The purpose of this paper is to formulate and investigate analogous statements for complex sequences.

Let \( \mathfrak{A} \) be the class of sequences, finite or infinite, \( \{a_k\} \) (denoted alternatively by \( A \)) of non-zero complex numbers for which \( \sum |a_k| < \infty \), and \( \{a^\prime\} \) (denoted alternatively by \( S \)), the general subsequence of \( \{a_k\} \) for fixed \( \{a_k\} \). Let \( \mathfrak{B} \) be the class of sequences \( \{b_k\} \) (denoted alternatively by \( B \)) of non-zero complex numbers for which \( \sum |b_k| = \infty \), and \( \{b^\prime\} \) (denoted alternatively by \( T \)), the general subsequence of \( \{b_k\} \) for fixed \( \{b_k\} \).

The following facts will be established: (i) Given any sequence \( \{a_k\} \in \mathfrak{A} \), there then exists a subsequence \( \{a^*\} \) for which \( |\Sigma a^\prime| = \sup |\Sigma a^\prime| \). (ii) If \( \rho = \inf_A \max_S |\Sigma a^\prime| / \sum |a_k| \), then \( \rho = 1/\pi \). (iii) No sequence \( \{a_k\} \in \mathfrak{A} \) exists for which \( \max_S |\Sigma a^\prime| / \sum |a_k| = \rho \). (iv) Given any sequence \( \{b_k\} \in \mathfrak{B} \), there exists a subsequence \( \{b^*\} \) such that‡

\[
\lim \sup |\Sigma b^\prime | / \sum |b_k| = \sup_T \lim N \sup |\Sigma b^\prime | / \sum |b_k|
\]

\[
= \lim N \sup |\Sigma b^\prime | / \sum |b_k| = \lim \sup_T \max |\Sigma b^\prime | / \sum |b_k|.
\]

(v) If \( \sigma = \inf_B \max_T \limsup_N |\Sigma b^\prime | / \sum |b_k| \), then \( \sigma = \rho \). (vi) There exists a sequence \( \{b_k\} \in \mathfrak{B} \) for which \( \max_T \limsup_N |\Sigma b^\prime | / \sum |b_k| = \sigma \).

Use will be made of abbreviations of the following sort: \( A_k \equiv |a_k| \), \( \phi_k \equiv \arg a_k \). For definiteness, the function “arg” will mean, throughout this paper, principal argument. Given any sequence \( \{a_k\} \in \mathfrak{A} \), define

\[
F(\phi) = \sum_{\cos(\phi - \phi_k) > 0} A_k \cos(\phi - \phi_k)
\]

\[
= \sum A_k \left\{ \cos(\phi - \phi_k) + |\cos(\phi - \phi_k)| \right\} / 2, \quad 0 \leq \phi \leq 2\pi.
\]

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‡ The notation \( \sum' \) indicates summation over precisely those elements of the subsequence which occur among the elements of the original sequence summed elsewhere in the formula.
Being continuous, \( F(\phi) \) attains its supremum. In what follows, to and including Theorem 3, \( \{a_k\} \) will signify an arbitrary but fixed sequence of class \( \mathscr{A} \).

**Theorem 1.** Let \( \phi^* \) be such that \( F(\phi^*) = \max F(\phi) \), and let \( \{a_k^*\} \) be the sequence of those elements of \( \{a_k\} \) for which \( \cos (\phi^* - \phi_k) > 0 \). Then \( \sup |\sum a_j| = F(\phi^*) = |\sum a_k^*| \).

**Proof.** Let \( \{a_j\} \) be any subsequence of \( \{a_k\} \), and define \( \phi = \arg \sum a_j \). Then

\[
|\sum a_j^*| \geq \sum A_j^* \cos (\phi^* - \phi_j) = F(\phi) \geq F(\phi).
\]

This establishes (i).

**Corollary 1.1.** In the notation of Theorem 1, \( \phi^* = \arg \sum a_k^* \).

**Proof.** Taking \( \{a_j\} = \{a_k^*\} \) in the inequalities of Theorem 1, we see that \( |\sum a_j^*| = \sum A_j^* \cos (\phi^* - \phi_j) \). That is, the modulus of \( \sum a_j^* \) is equal to that of its projection on the ray of angle \( \phi^* \).

The following theorem and its corollary provide a sort of converse or dual of Theorem 1 and Corollary 1.1:

**Theorem 2.** Let \( \{\bar{a}_j\} \) be a subsequence of \( \{a_k\} \) for which \( |\sum \bar{a}_j| = \max |\sum a_j| \), and let \( \bar{\phi} = \arg \sum \bar{a}_j \). Then \( \max F(\phi) = |\sum \bar{a}_j| = F(\bar{\phi}) \).

**Proof.** Let \( \phi \) be any angle, \( 0 \leq \phi \leq 2\pi \), and \( \{a_j\} \) the sequence of those elements of \( \{a_k\} \) for which \( \cos (\phi - \phi_k) > 0 \). Then

\[
F(\phi) = \sum A_k \cos (\phi - \phi_k) \geq |\sum \bar{A}_j \cos (\bar{\phi} - \bar{\phi}_j) = |\sum \bar{a}_j| \geq \sum A_k \cos (\phi - \phi_k) = F(\phi).
\]

**Corollary 2.1.** In the notation of Theorem 2, \( \{\bar{a}_j\} \) is the sequence of those elements of \( \{a_k\} \) for which \( \cos (\bar{\phi} - \phi_k) > 0 \).

**Proof.** Taking \( \phi = \bar{\phi} \) in the inequalities of Theorem 2, we see that

\[
\sum A_k \cos (\phi - \phi_k) = \sum \bar{A}_j \cos (\bar{\phi} - \bar{\phi}_j).
\]

In conjunction with Theorem 3 (below), this proves the assertion.

**Theorem 3.** In the notation of Theorem 2, there exists no element \( a_k \) of \( \{a_k\} \) for which \( \cos (\bar{\phi} - \phi_k) = 0 \).
PROOF. If there were such an element, then \(|\sum a_i \pm a_k| > |\sum a_i|\), so that addition of \(a_k\) to \(\{a_i\}\), if it were not already therein contained, or removal of it, if it were, would provide a subsequence of \(\{a_k\}\) to establish that \(|\sum a_i| < \max_s |\sum a_i|\), contrary to the definition of \(\{\tilde{a}_i\}\).

**Theorem 4.** \(\rho = 1/\pi\).

**Proof.** First,

\[
\int_0^{2\pi} F(\phi) d\phi = 2 \sum A_k.
\]

Thus max \(F(\phi) \geq \sum A_k/\pi\), whence, by Theorem 1, \(\rho \geq 1/\pi\). To show that \(\rho \leq 1/\pi\), consider the sequence over \(\nu\) of particular finite sequences \(\{a_k\}\), where \(a_k = \exp \{k\pi i/(2\nu + 1)\}\), \((k = -2\nu, -2\nu + 1, \cdots, 0, 1, \cdots, 2\nu, 2\nu + 1)\). By Corollary 2.1, for given \(\nu\) any subsequence \(\{a_i\}\) of \(\{a_k\}\) the sum of whose elements is of maximum modulus consists of those elements whose arguments lie in a certain sector of aperture \(\pi\). By the symmetry of the sequence \(\{a_k\}\), the midray of such a sector must lie either on a vector \(a_k\) or midway between two such vectors which are adjacent. In the latter case, however, Theorem 3 would be violated. Hence the former must obtain, and thus those elements of \(\{a_k\}\) for which \(-\pi/2 < k\pi/(2\nu + 1) < \pi/2\) constitute a subsequence the sum of whose elements is of maximum modulus. Hence, if \(S(\nu)\) denotes the general subsequence \(\{a_i\}\) of \(\{a_k\}\),

\[
\max_{S(\nu)} \left| \sum_j x_a_j \right| / \sum_k x_A_k = \sum_{h=-\nu}^{\nu} \cos \left\{ h\pi/(2\nu + 1) \right\} / \left\{ 2(\nu + 1) \right\} = 1 / \left\{ 2(\nu + 1) \right\} \sin \left\{ \pi / \left\{ 2(\nu + 1) \right\} \right\} ;
\]

and, as \(\nu \to \infty\), this tends monotonically to \(1/\pi\). This establishes (ii).

**Theorem 5.** There exists no sequence \(\{a_k\} \in \mathbb{A}\) for which \(F(\phi)\) is constant.

**Proof.** If there were such a sequence \(\{a_k\}\) then, by Theorem 1, for each \(\phi\) the sequence \(\{a_j^{*}\}\) of those elements of \(\{a_k\}\) for which \(\cos (\phi - \phi_k) > 0\) would be such that \(|\sum a_j^{*}| = \max_s |\sum a_i|\). Hence, by Corollary 1.1 and Theorem 3, there would exist no non-zero element of \(\{a_k\}\), contrary to the definition of \(\mathbb{A}\).

**Theorem 6.** Given an arbitrary sequence, finite or infinite, of pairs \((C_k, \psi_k)\), where the \(\psi_k\) are real numbers and the \(C_k\) positive numbers with \(\sum C_k < \infty\), then \(\Phi(\phi) = \sum C_k |\cos (\phi - \psi_k)|\) is not constant.
The sequence \( \{a_k\} \) defined thus: \( a_{2k-1} = C_k \exp(ik) \), \( a_{2k} = C_k \exp\left[i(\psi_k - \pi)\right] \), is of class \( \mathcal{A} \), and

\[
F(\phi) = \sum_{\cos(\phi - \psi_k) > 0} C_k \cos(\phi - \psi_k) + \sum_{\cos(\phi - \psi_k) < 0} C_k \cos(\phi + \pi - \psi_k)
\]

\[
= \sum C_k \left| \cos(\phi - \psi_k) \right| = \Phi(\phi).
\]

The conclusion now follows from Theorem 5.

**Theorem 7.** There exists no sequence \( \{a_k\} \in \mathcal{A} \) for which it is true that \( \max_\phi |a_{\phi}/A_k| = \rho \).

**Proof.** If there were such a sequence \( \{a_k\} \), then, by Theorem 1, 

\[
F(\phi) \leq \rho \sum A_k \quad \text{for all } \phi.
\]

Hence

\[
\int_{0}^{2\pi} \rho \sum A_k - F(\phi) \, d\phi = \int_{0}^{2\pi} \left\{ \rho \sum A_k - F(\phi) \right\} d\phi = 2 \sum A_k - 2 \sum A_k = 0.
\]

By continuity, then, \( F(\phi) = \rho \sum A_k \) for all \( \phi \). But by Theorem 5 this is impossible. This establishes (iii).

**Lemma 8.1.** Let \( X \) be an aggregate of elements \( x \) of any sort, and \( \{f_N\} \) any sequence of functionals over \( X \). Then \( \sup_x \limsup_\phi f_N(x) \leq \limsup \sup_x f_N(x) \).

**Proof.** For each \( N \) and for all \( x, f_N(x) \leq \sup_x f_N(x) \). Hence, for all \( x \), \( \limsup \sup_x f_N(x) \leq \limsup \sup_x f_N(x) \), and the conclusion follows.

**Remark.** Equality in the conclusion of Lemma 8.1 is not implied by the hypotheses. For, if we let \( X \) represent the totality of rational numbers and define \( f_N(1/N) = 1, f_N(x) = 0 \) for \( x \neq 1/N, (N = 1, 2, \ldots) \), it follows that \( \limsup f_N(x) = 0 \) for each \( x \), so that \( \sup_x \limsup f_N(x) = 0 \), whereas \( \sup_x f_N(x) = 1 \) for each \( N \), so that \( \limsup \sup_x f_N(x) = 1 \).

**Theorem 8.** Let \( \{b_k\} \in \mathcal{B} \) be arbitrary. Then there exists a subsequence \( \{b_{k}^{*}\} \) of \( \{b_k\} \) for which

\[
\limsup N \sum_{1}^{N} b_{k}^{*} \bigg/ \sum_{1}^{N} b_{k} = \limsup T \sum_{1}^{T} b_{k} = \limsup N \sum_{1}^{N} b_{k}^{*} \bigg/ \sum_{1}^{N} b_{k}.
\]

**Proof.** By Theorem 1, for each \( N \) there exists a subsequence \( \{b_{k}^{(N)}\} \) of \( \{b_k\} \) for which \( \left| \sum_{1}^{N} b_{k}^{(N)} \right| \bigg/ \sum_{1}^{N} b_{k} = \sup_T \left| \sum_{1}^{T} b_{k} \right| \bigg/ \sum_{1}^{T} b_{k} \).

Let \( \{N(v)\}, (v = 1, 2, \ldots) \), be a subsequence of \( \{N\} \) such that
\[
\lim \frac{\sum_{i=1}^{N(v)} b_i}{\sum_{i=1}^{N(v)} B_k} = \limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^{(N)}}{\sum_{i=1}^{N} B_k},
\]
and such that
\[
\sum_{i=1}^{N(v)-1} B_k / \sum_{i=1}^{N(v)} B_k < 1/2^{v+1},
\]
where the notation \(b_i\) represents \(b_i^{(N)}\) with \(N = N(v)\). Define the subsequence \(\{b_i^*\}\) of \(\{b_i\}\) in such a manner that its elements coincide in order with those of \(\{b_j\}\) in the subscript interval (with respect to the original sequence \(\{b_k\}\) \(N(v-1) < k \leq N(v)\) for all \(v\), \((N(0) = 0)\). Now
\[
\limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^*}{\sum_{i=1}^{N(v)} B_k} \geq \limsup_{v} \frac{\sum_{i=1}^{N(v)} b_i}{\sum_{i=1}^{N} B_k},
\]
so that from the inequality
\[
\left| \sum_{i=1}^{N(v)} b_i^* \right| / \sum_{i=1}^{N(v)} B_k \geq \left| \sum_{i=1}^{N(v)} b_i \right| / \sum_{i=1}^{N} B_k - 1/2^v,
\]
it follows that
\[
\limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^*}{\sum_{i=1}^{N(v)} B_k} \geq \limsup \max_{T} \left| \sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k \right|
\]
But that
\[
\limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^*}{\sum_{i=1}^{N(v)} B_k} \leq \sup_{T} \limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k}{\max_{T} \left| \sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k \right|}
\]
is obvious, and that
\[
\sup_{T} \limsup_{N} \frac{\sum_{i=1}^{N(v)} b_i^*}{\sum_{i=1}^{N(v)} B_k} \leq \limsup_{N} \max_{T} \left| \sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k \right|
\]
follows from Lemma 8.1. The conclusion follows. This establishes (iv).

**Lemma 9.1.** \(\sigma \geq \rho\).

**Proof.** By Theorem 8,
\[
\sigma = \inf_B \max_T \limsup_N \frac{\sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k}{\sum_{i=1}^{N(v)} B_k}
\]
\[
= \inf_B \limsup_N \max_T \frac{\sum_{i=1}^{N(v)} b_i^* / \sum_{i=1}^{N} B_k}{\sum_{i=1}^{N} B_k} \geq \rho.
\]
which establishes the lemma.

Consider now the sequence \( \{ b_k^* \} \) defined thus: \( b_k^* = e^{ik} \), \( (k = 1, 2, \cdots) \), and define

\[
F_N(\phi) = \sum_{k=1}^{N} \left\{ \cos (\phi - k) + | \cos (\phi - k) | \right\} / 2N, \quad 0 \leq \phi \leq 2\pi.
\]

**Lemma 9.2.** \( \lim_{N} \text{osc}_\phi F_N(\phi) = 0. \)

**Proof.** Let \( \epsilon > 0 \) be arbitrary; let \( K \) be such that, for each \( \phi \), \( \phi \equiv \rho(\phi) + \eta(\mod 2\pi) \) for some \( \eta(\phi) \) for which \( |\eta(\phi)| < \epsilon \) and some integer \( \rho(\phi) \) for which \( 0 \leq \rho(\phi) \leq K \); and let \( N \) be such that \( K/N < \epsilon \). Then, for each \( \phi \),

\[
|F_N(\phi) - F_N(0)| \leq \left| \sum_{k=1}^{N-p(\phi)} \left\{ \cos (k - \eta(\phi)) + \cos (\phi - k) \right\} / 2N + \sum_{k=p(\phi)+1}^{N} \left\{ \cos k + | \cos k | \right\} / 2N < 3\epsilon.
\]

This establishes the lemma.

**Lemma 9.3.** \( \lim_{N} F_N(\phi) = \rho \) uniformly in \( \phi \).

**Proof.** The assertion follows from Lemma 9.2 and the fact that, for each \( N, \int_{0}^{\phi} F_N(\phi)d\phi = 2. \)

**Theorem 9.** \( \sigma = \rho. \)

**Proof.** Applying Theorem 2 to the (finite) sequence of those elements of \( \{ b_k^* \} \) for which \( k \leq N \), we find that

\[
\max_T | \sum_{i} b_i^* / \sum_{i} B_i^* = \max_{\phi} F_N(\phi),
\]

which tends to \( \rho \), by Lemma 9.3. By Theorem 8 and Lemma 9.1, this establishes the theorem, and hence also (v).

**Theorem 10.** There exist an uncountably infinite number of subsequences \( \{ b_k^*_r \} \) of \( \{ b_k^* \} \) for which

\[
\lim_{N} \left| \sum_{i} b_i^*_r / \sum_{i} B_i^* \right| = \max_{T} \limsup_{N} \left| \sum_{i} b_i^*_r / \sum_{i} B_i^* \right| = \rho = \sigma.
\]
PROOF. Let \( \phi' \) be arbitrary, and let \( \{b_k^*\} \) be the sequence of those elements of \( \{b_k^*\} \) for which \( \cos (\phi' - \phi_k^*) > 0 \). Then, by inequalities like those used in the proof of Theorem 2, for each \( N \),

\[
F_N(\phi') \leq \left| \sum_{i=1}^{N} b_i^* \right| / \sum_{i=1}^{N} B_i^* \leq \max_{\phi} \left| \sum_{i=1}^{N} b_i^* \right| / \sum_{i=1}^{N} B_i^* = \max_{\phi} F_N(\phi),
\]

and the conclusion is seen to follow from Lemma 9.3 and Theorem 8. This establishes (vi).

**Theorem 11.** If \( \Phi_N(\phi) = \sum_{i=1}^{N} \cos (\phi - k) \), \( (0 \leq \phi \leq 2\pi) \), then \( \lim_{N} \Phi_N(\phi) = 2/\pi \) uniformly in \( \phi \).

**Proof.** As in the proof of Lemma 9.2, it can be shown that \( \lim_{N} \cos \phi \Phi_N(\phi) = 0 \). Also,

\[
\int_{0}^{2\pi} \Phi_N(\phi) d\phi = 4.
\]

The conclusion follows.

**Remark.** The sequence \( \{b_k^*\} \) could equally well have been taken thus: \( b_k^* = e^{ik\delta} \), \( (k = 1, 2, \cdots) \), where \( \delta \) is any number incommensurable with \( \pi \).

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