CARATHÉODORY ON GEOMETRICAL OPTICS

Geometrische Optik. By C. Carathéodory. (Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 5.) Berlin, Springer, 1937. 4+104 pp.

As a formal introduction to the mathematical analysis of the principles of geometrical optics this comparatively slender volume, comprising little more than a hundred pages, is a contribution of unusual value that should be welcomed by all students of the subject. The author's qualifications for his task are undoubted. On the other hand, in order to read this purely mathematical treatise with advantage the student must be qualified too by a due initiation in the realms of both geometry and optics. The author rightly considers that the application of the theory in the design and construction of optical instruments lies wholly outside the scope of his book. The basis of his whole argument is indeed derived from the brilliant speculations as to the nature of light of two great contemporary philosophers of the latter half of the seventeenth century, Pierre Fermat (1601–1665) and Christian Huygens (1629–1695). According to the former *"la nature agit toujours par les voies les plus courtes"* (principle of the quickest route or least time), whereas, according to Huygens's "principle of accumulated disturbances," the effect at any place in the all-pervading luminiferous ether is to be regarded as the resultant of the innumerable partial or secondary influences that concur there from all directions (method of construction of the enveloping wave-front). Starting from these two general principles of the propagation of light (although as a matter of fact Huygens's principle may be deduced from Fermat's), the author of this monograph on geometrical optics regards his task as finished when he has developed the laws of optical imagery of the first order (as Gauss did for a symmetrical optical instrument in 1841), not attempting therefore to pursue the intricate investigation of the aberrations of the third order.

The seventeenth century was indeed a notable epoch in optical science. It was ushered in, so to speak, by the invention of the telescope, inseparably associated with the names of G. Galilei (1564–1642), J. Kepler (1571–1630), and C. Scheiner (1575–1650), to be followed so soon by the inventions of the compound microscope and the magic lantern. The year after Galileo died Sir Isaac Newton (1643–1727) was born, and thus the entire century was contained within the lifetimes of these two extraordinary men. In 1621 an eminent Dutch professor in the University of Leyden named Willebrod Snell (1581–1626) had succeeded at last in finding the elusive law of the refraction of light, but he died without publishing the result; and so the sine law of refraction was not generally known until it was first publicly announced by René Descartes (1596–1650) in his *Dioptrics*, which appeared in 1637. Entirely ignorant of Snell's earlier investigations, this great philosopher had derived the law independently in consequence of his theory that light was a pressure transmitted through the so-called plenum of space. At that time it had not yet been established that light travelled with a finite velocity; nevertheless, according to Descartes' hypothesis its speed would have to be greater in the denser of two media, exactly contrary to the inference afterwards drawn from the wave theory of light as developed by Huygens.

In the two decades that succeeded the publication of Descartes' book, the sine law of refraction was completely verified by experiment. As early as 1657 Fermat had sought to formulate a general principle or law of economy for the transmission of light as the foundation of the science of dioptrics. However, inasmuch as his conception was diametrically opposed to the theory of Descartes, Fermat hesitated to pursue his speculations for fear that they would necessarily lead to conclusions that were inconsistent with the ascertained law of refraction. Accordingly, some years later, when he was induced to resume the study of the question (1661), he was much surprised and gratified at the same time to find that as a matter of fact his minimum hypothesis led to precisely the same law of refraction as Descartes had deduced from totally different assumptions. However, in spite of this encouragement Fermat continued to be perplexed by some minor difficulties in the way of his theory, because examples could be adduced in which the route taken by the light, instead of being the quickest, was on the contrary the longest of any. This particular difficulty was pointed out again long afterwards by Sir William Rowan Hamilton (1805–1865) in the following passage taken from one of his famous papers published in 1833:

"If an eye be placed in the interior but not at the centre of a reflecting hollow sphere, it may see itself reflected in two opposite points, of which one indeed is the nearest to it, but the other on the contrary is the furthest; so that of the two different paths of light, corresponding to these two opposite points, one indeed is the shortest, but the other is the longest of any. In mathematical language, the integral called action, instead of being always a minimum, is often a maximum; and often it is neither the one nor the other, though it has always a certain stationary property, · · · ." (Hamilton's *Mathematical Papers*, vol. 1, edited by A. W. Conway and J. L. Synge, Cambridge, 1931, p. 318.)

According to Huygens' wave theory, the optical length between two points in the path of a ray of light is equal to the distance that would have been traversed if the disturbance had been propagated with the constant velocity of light *in vacuo*; the analytical expression of this distance being therefore the integral $\int n\cdot dl$, where n denotes the (constant or variable) index of refraction of the actual optical medium and dl denotes an element of arc of the trajectory. Now Fermat made the mistake of assuming that this integral was invariably a minimum function. Had the calculus of variations been invented then, doubtless Fermat, keen mathematician that he was, would have formulated his principle correctly by saying that the necessary and sufficient condition that a linear path shall be a possible route of a ray of light is that the first variation of the optical length of any portion of the trajectory must be vanishingly small, that is, $\delta\int n\cdot dl=0$. In other words, to use Hamilton's mode of expression, the true theory is a principle of stationary action rather than of least action.

The fact that a normal congruence of rays remains a normal congruence after refraction or reflection, as announced by E. L. Malus (1775–1812) in 1808, that is, the fact that the wave surface is a surface of stationary action which is therefore cut orthogonally by rays of light emanating originally from a point-source, may be deduced immediately from Fermat's principle.

All these questions, including also a preliminary reference to Hamilton's characteristic function and a brief discussion of Descartes' aplanatic optical surfaces of revolution ($n\cdot l\pm n'\cdot l'=$constant), are competently treated in the first two chapters of Carathéodory's book. The three remaining chapters which follow in logical order are devoted to the theory of optical imagery in its various aspects, comprising such subjects as the so-called "Eikonal" functions of H. Bruns (1848–1919), Maxwell's "fish-eye," and Gauss's theory.

After all is said and done, Hamilton's classical papers published about a century ago in the Transactions of the Royal Irish Academy contain the complete theory of systems of rays of light. This masterly work remains for all time, but Hamilton's writings, it must be admitted, are not easy reading and for that reason are not generally known even to this day. In the opinion of the present reviewer, not the least merit of Carathéodory's book on geometrical optics is that it is a comparatively easy means of access to Hamilton's "grand theory," as it was called by P. G. Tait (1831–1901) in his essay on *Light* in the ninth edition of the Encyclopaedia Britannica.

J. P. C. SOUTHALL

VOLTERRA AND PÉRÈS ON FUNCTIONALS

Théorie Générale des Fonctionnelles. Vol. 1. By Vito Volterra and Joseph Pérès. Paris, Gauthier-Villars, 1936. 12+357 pp.

The book at hand is the first of a sequence of three volumes entitled *Théorie Générale des Fonctionnelles*, written jointly by Volterra and Pérès. The present volume was published in the Borel series of monographs on the theory of functions. The second and third volumes have not been published as yet.

Volterra was one of the pioneers in the study of the theory of functionals. In 1913 he published in the Borel series his first book on the subject. It was entitled *Leçons sur les Fonctions des Lignes* and was reviewed by Professor Bliss in this Bulletin, vol. 21 (1915), pp. 345–355. In 1924 Volterra and Pérès published jointly a book on composition of functions and permutable functions. In 1926 Volterra gave a set of lectures on the theory of functionals at Madrid. These lectures were published in book form in Spanish. A translation and revision of them was given in 1930 in a book entitled *Theory of Functionals*. This book is descriptive in character and contains no detailed proofs of the subject matter discussed. The purpose of the three volumes, of which the present volume is the first, is to give a systematic and detailed study of the material found in the books described above, particularly the last. The authors, of course, will incorporate as much as possible of the newer developments in the theory of functionals.

The subject matter to appear in the three volumes is to be divided as follows. The first volume is devoted to the general theory of functionals and to the theory of integral equations. In the second volume the authors will develop the theory of composition of functions and its relation to integral equations, the theory of integro-differential equations, and extensions of the theory of analytic functions. In the third volume they will be concerned chiefly with the completion of the theories developed in the first two volumes and with applications of these results. In particular, they will discuss the modern theories of the calculus of variations and of analytic functionals and their applications to mechanics, to mathematical physics, to biology, to statistics and political economy.

The present volume is divided into two parts. In the first five chapters the authors develop a general theory of functionals, particularly functionals of curves. In the remaining six chapters they are concerned chiefly with the theory of integral equations. Chapter I is introductory in character and contains principally a development of the concept of functionals. Here and elsewhere the authors give numerous examples. Chapter II is devoted to the study of elementary properties of metric spaces and to the concepts of continuity and semicontinuity. The theory of the Lesbesgue integral is developed. The method used is a modification of one given by Riesz. In Chapter III the authors discuss linear functionals of curves and also homogeneous functionals of higher degree. The representation theorems of Hadamard and Riesz