writings, it must be admitted, are not easy reading and for that reason are not generally known even to this day. In the opinion of the present reviewer, not the least merit of Carathéodory's book on geometrical optics is that it is a comparatively easy means of access to Hamilton's "grand theory," as it was called by P. G. Tait (1831–1901) in his essay on *Light* in the ninth edition of the Encyclopaedia Britannica.

J. P. C. SOUTHALL

VOLTERRA AND PÉRÈS ON FUNCTIONALS

Théorie Générale des Fonctionnelles. Vol. 1. By Vito Volterra and Joseph Pérès. Paris, Gauthier-Villars, 1936. 12+357 pp.

The book at hand is the first of a sequence of three volumes entitled *Théorie Générale des Fonctionnelles,* written jointly by Volterra and Pérès. The present volume was published in the Borel series of monographs on the theory of functions. The second and third volumes have not been published as yet.

Volterra was one of the pioneers in the study of the theory of functionals. In 1913 he published in the Borel series his first book on the subject. It was entitled *Leçons sur les Fonctions des Lignes* and was reviewed by Professor Bliss in this Bulletin, vol. 21 (1915), pp. 345–355. In 1924 Volterra and Pérès published jointly a book on composition of functions and permutable functions. In 1926 Volterra gave a set of lectures on the theory of functionals at Madrid. These lectures were published in book form in Spanish. A translation and revision of them was given in 1930 in a book entitled *Theory of Functionals.* This book is descriptive in character and contains no detailed proofs of the subject matter discussed. The purpose of the three volumes, of which the present volume is the first, is to give a systematic and detailed study of the material found in the books described above, particularly the last. The authors, of course, will incorporate as much as possible of the newer developments in the theory of functionals.

The subject matter to appear in the three volumes is to be divided as follows. The first volume is devoted to the general theory of functionals and to the theory of integral equations. In the second volume the authors will develop the theory of composition of functions and its relation to integral equations, the theory of integro-differential equations, and extensions of the theory of analytic functions. In the third volume they will be concerned chiefly with the completion of the theories developed in the first two volumes and with applications of these results. In particular, they will discuss the modern theories of the calculus of variations and of analytic functionals and their applications to mechanics, to mathematical physics, to biology, to statistics and political economy.

The present volume is divided into two parts. In the first five chapters the authors develop a general theory of functionals, particularly functionals of curves. In the remaining six chapters they are concerned chiefly with the theory of integral equations. Chapter I is introductory in character and contains principally a development of the concept of functionals. Here and elsewhere the authors give numerous examples. Chapter II is devoted to the study of elementary properties of metric spaces and to the concepts of continuity and semicontinuity. The theory of the Lesbesgue integral is developed. The method used is a modification of one given by Riesz. In Chapter III the authors discuss linear functionals of curves and also homogeneous functionals of higher degree. The representation theorems of Hadamard and Riesz are given. Operations on functionals of curves including differentials and derivatives are taken up in Chapter IV. An extension of Taylor's theorem is given in terms of these derivatives and differentials. A theory of integration of functionals is also developed. In the introductory sections of Chapter V the authors point out relationships between the calculus of variations and the theory of functionals. The remainder of the chapter was written by Tonelli and contains a clear exposition of the methods of Tonelli used in establishing existence theorems. Existence theorems for absolute minima of simple integrals are given.

The second part of the book is devoted to the inverse problem, that is, to the problem of finding a function $y(t)$ satisfying the functional equation

$$F\,[y(t_a^b),x] = z(x), \tag{1}$$

where F and z are given. The authors restrict themselves for the most part to integral equations. In Chapters VI and VII is found an excellent discussion of Volterra equations of the first and second kinds. The authors consider in Chapter VII equations with non-bounded kernels, equations whose interval of integration is infinite, and equations of the first kind whose kernels have isolated zeros on the diagonal. They also take up "integral-functional equations." A simple example of these is the equation

$$\phi(y) - P(y)\,\phi(\alpha y) - \lambda\int_0^y \phi(x)\,K(x,y)\,dx = h(y),$$

where α is a constant, P, K, h are known functions, and $\phi(y)$ is the unknown function. The method of successive approximations is used here. Equations of Volterra with two variable limits are also discussed.

An excellent treatment of Fredholm equations of the first and second kinds is given in Chapters VIII, IX, and X. In Chapter IX the authors take up, among other things, kernels $K(x,y)$ for which $\int_a^b |K(x,y)|\,dy$ or $\int_a^b |K(y,x)|\,dy$ is uniformly bounded for almost all x, and also kernels of the form $H(x,y)/(y-x)^\alpha$, where $H(x,y)$ is continuous. They treat next singular cases, such as the equation of Picard,

$$A(x)\,\phi(x) + \lambda\int_a^b K(x,y)\,\phi(y)dy = f(x),$$

where $A(x)$ vanishes at some point on $a \leqq x \leqq b$. They consider also equations in which the principal values of the integrals are used. In Chapter X the authors discuss symmetric and symmetrizable kernels, expansion theorems, the theorem of Hilbert, the Riesz-Fisher theorem, and fundamental functions of Schmidt and Fredholm equations of the first kind.

Non-linear integral equations are studied in the last chapter. Equations of the forms

$$\phi(x) - \int_a^x H(x,y,\,\phi(y))dy = h(x), \quad \phi(x) - \int_a^b H(x,y,\,\phi(y))\,dy = h(x)$$

are discussed first. Then a generalized functional determinant of equation (1) above is defined and a general implicit function theorem is stated. The theorem is justified in special cases. The remainder of the chapter is devoted to equations of Schmidt and extensions of these equations.

At the end of the book one finds an extensive bibliography. The book is well written and is a valuable addition to mathematical literature. This volume, together with the two which are yet to be published, undoubtedly will be especially useful as a reference for the theory of functionals.

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