A THEOREM ON INTERIOR TRANSFORMATIONS*

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In an earlier paper† it has been shown that if $T(A)=B$ is a light interior transformation, that is, a continuous transformation mapping open sets into open sets and mapping no continuum into a single point, and if A is compact, then for every simple arc pq in B and any $p_0 \epsilon T^{-1}(p)$ there exists a simple arc p_0q_0 in A which maps topologically onto pq under T.

In this paper the following extension to arbitrary dendrites (that is, locally connected continua containing no simple closed curves) in B will be made.

THEOREM. *Let $T(A)=B$ be interior and light, where A is compact. For any dendrite D in B and any $x_0 \epsilon T^{-1}(D)$ there exists a dendrite E in A containing x_0 which maps topologically onto D under T.*

PROOF. Since the transformation $TT^{-1}(D)=D$ is interior,‡ clearly there is no loss of generality in assuming that $D=B$. Now let $H=\sum p_iq_i$ be an arc-development of B; that is, for each $i>0$, $p_iq_i \sum_{j=1}^{j=i-1} p_jq_j = q_i$, and p_i, $(i=1, 2, \cdots)$, and all points of $\overline{H}-H$ are end points of B, where we may suppose that $p_0=T(x_0)$. Now by the result quoted above, there exists an arc x_0y_0 such that $T(x_0y_0)=p_0q_0$ and T is topological on x_0y_0. Likewise there exists an arc x_1y_1 in A such that $T(x_1y_1)=p_1q_1$, T is topological on x_1y_1, and furthermore so that $y_1=T^{-1}(q_1)\cdot x_0y_0$. Similarly there is an arc x_2y_2 contained in A so that $T(x_2y_2)=p_2q_2$, T is topological on x_2y_2, and $y_2=T^{-1}(q_2)\cdot\sum_0^1 x_iy_i$. Continuing this process indefinitely we obtain $K=\sum x_iy_i$, so that $T(K)=H$, and T is topological on K. Clearly $\overline{K}$ is a continuum and $T(\overline{K})=B$. Hence our proof will be complete as soon as we show that T is 1 to 1 on $\overline{K}$, or, what amounts to the same thing, that for each $p \epsilon B$, $T^{-1}(p)\cdot\overline{K}$ reduces to a single point.

Now there exists a monotone decreasing sequence of connected neighborhoods $V_1, V_2, V_3, \cdots$ of p in B with $\delta(V_i)\rightarrow 0$. Furthermore,

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† See my paper in the Duke Mathematical Journal, vol. 3 (1937), p. 377, Theorem 4.1. Compare with Stoilow, Annales Scientifiques de l'École Normale Supérieure, vol. 63 (1928), pp. 347–382; and Montgomery, Transactions of this Society, vol. 42 (1937), pp. 328–329.

‡ See my paper, loc. cit., p. 370, Lemma 1.2.

since $\overline{H}-H$ contains only end points of B, it follows that $H\cdot V_1$, $H\cdot V_2, \cdots$ are connected sets which, of course, are open in H. Hence $K\cdot T^{-1}(V_1)$, $K\cdot T^{-1}(V_2), \cdots$ is a monotone decreasing sequence of connected sets in K which are open in K. Then let $L=\lim K\cdot T^{-1}(V_i)$. Since $V_i\to p$, we have $L\subset T^{-1}(p)$. But, since L is necessarily connected and $T^{-1}(p)$ is totally disconnected, L must reduce to a single point $q\epsilon T^{-1}(p)$. Hence $T^{-1}(p)\cdot\overline{K}=q$, and our theorem is established.

That a similar conclusion cannot be obtained, if we permit D to contain simple closed curves, is readily seen, since it fails to hold even in the simple transformation $w=z^2$ of the circle $|z|=1$ into the circle $|w|=1$. Still more striking, however, is the following:

EXAMPLE. *There exist two connected graphs A and B and an interior transformation $T(A)=B$, where A contains no subset homeomorphic with B.*

Let $J=a_1b_1f_1c_1a_2b_2f_2c_2$ be a simple closed curve in a plane, where the points $a_1, b_1, \cdots$ are cyclically ordered on J as indicated. Let $a_1d_1f_2$ and $b_1e_1c_1$ be disjoint arcs lying within J except for their end points, and let d_1e_1 be an arc within J having only its end points in common with $a_1d_1f_2+b_1e_1c_1$. Similarly, let $f_1d_2a_2$ and $b_2e_2c_2$ be disjoint arcs lying, except for their end points, without J, and let d_2e_2 be an arc without J having just its end points in common with $f_1d_2a_2+b_2e_2c_2$. Finally let A be the graph thus constructed, that is,

$$\begin{aligned}A &= J + a_1d_1f_2 + b_1e_1c_1 + d_1e_1 + f_1d_2a_2 + b_2e_2c_2 + d_2e_2\\ &= a_1b_1 + b_1f_1 + f_1c_1 + c_1a_2 + b_1e_1 + e_1c_1 + a_1d_1 + d_1f_2 + d_1e_1\\ &\quad + a_2b_2 + b_2f_2 + f_2c_2 + c_2a_1 + b_2e_2 + e_2c_2 + a_2d_2 + d_2f_1 + d_2e_2.\end{aligned}$$

Then A is a graph of eighteen edges as indicated in the latter sum.

We now construct a graph B in 3-space as follows: Let θ be a θ-curve $bac+bec+bfc$ in a plane π. Let d be a point not in π, and let da, de, and df be arcs having just d in common and having just a, e, and f, respectively, in common with θ.

If we now let

$$\begin{aligned}B &= \theta + da + de + df\\ &= ab + bf + fc + ca + be + ec + ad + df + de,\end{aligned}$$

then B is a graph of nine edges as here indicated.

Now let $T(A)=B$ be the transformation which maps a_1b_1, a_2b_2 topologically into ab, b_1f_1, and b_2f_2 topologically into bf, and so on. In general T maps any edge x_iy_j of A topologically, preserving end points, into the edge xy of B. Then T is an interior transformation. In fact T is a local homeomorphism which is 2 to 1.

Since A is a planar graph, whereas B is non-planar (B is, in fact, one of the two well known Kuratowski primitive skew curves), clearly A contains no subset homeomorphic with B. Incidentally this example shows that *planarity is not an interior property* (that is, it is not invariant under interior transformations).

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ON THE TRANSFORMATION GROUP FOR DIABOLIC MAGIC SQUARES OF ORDER FOUR*

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This paper concerns only magic squares of order four, and all statements of the paper are to be construed as applying only to magic squares of order four.

One says that

(1)

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

is a diabolic (or pan-diagonal or Nasik) magic square if $a, b, \cdots, p$ are $1, 2, \cdots, 16$ in some order, and each row, column, and diagonal adds up to 34. This is to include broken diagonals such as i, f, c, p, or c, h, i, n. A diabolic magic square clearly remains diabolic if subjected to the following transformations:

A. Reflection about the a, f, k, p diagonal.
B. Rotation through 90° counter-clockwise.
C. Putting the first column last.
D. Putting the first row last.

For many purposes it is convenient to consider a diabolic magic

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