A NOTE ON FREDHOLM-STIELTJES INTEGRAL EQUATIONS*

F. G. DRESSEL

1. **Introduction.** The object of this paper is to show that the integral equation†

$$(1) \qquad f(x) = m(x) + \lambda \int_0^1 f(y) dG(x, y), \qquad 0 \leqq x, y \leqq 1,$$

can be changed into an ordinary Fredholm equation when $G(x, y)$ is absolutely continuous $g(y)$.‡ The integration is carried out in the Young-Stieltjes sense, and $g(y)$ is a bounded, monotone increasing function.

2. **Lemmas.** If $h(x)$ is of bounded variation and we set $h(x) = h(0)$, $(x < 0)$, and $h(x) = h(1)$, $(x > 1)$, then we may define the completely additive function of sets $\bar{h}(e)$ by

$$\bar{h}(e) = h(x_2 + 0) - h(x_1 - 0), \qquad e = e(x_1 \leqq t \leqq x_2).$$

Using this notation we have the following lemma:

LEMMA 1. *If $f(x)$ is measurable Borel then*

$$\int_0^1 f(x) dh(x) = \int_0^1 f(x) d\bar{h},$$

the left side being Young-Stieltjes integration, the right Radon-Stieltjes.

In case one integral does not exist the equality sign is taken to mean that the other integration is non-existent. Because of the properties of the integrals under consideration, we need only prove the equality for the functions

$$f_1(x) = 1, \; x = \alpha, \qquad f_2(x) = 1, \; 0 \leqq \alpha < x < \beta \leqq 1,$$
$$= 0, \; x \neq \alpha; \qquad = 0, \text{ elsewhere}.$$

* Presented to the Society, December 29, 1936.

† For a discussion of (1) see G. C. Evans and O. Veblen, *The Cambridge Colloquium Lectures on Mathematics*, American Mathematical Society Colloquium Publications, vol. 5, 1922, p. 101.

‡ For terminology see Alfred J. Maria, *Generalized derivatives*, Annals of Mathematics, vol. 28 (1926–1927), pp. 419–432. I am much indebted to Mr. Maria for many valuable suggestions.

All functions used in the present paper are assumed to be measurable Borel.

We have

$$\int_0^1 f_1(x)dh(x) = h(\alpha+0) - h(\alpha-0) = \bar{h}(\alpha) = \int_0^1 f_1(x)d\bar{h};$$

$$\int_0^1 f_2(x)dh(x) = h(\beta-0) - h(\alpha+0) = \bar{h}(e) = \int_0^1 f_2(x)d\bar{h},$$

where e is the open set $\alpha<t<\beta$.*

LEMMA 2. *If $G(x)$ is absolutely continuous with respect to the bounded monotone increasing function $g(x)$, then*

$$\int_0^1 f(x)dG(x) = \int_0^1 f(x)DG(x)dg(x),$$

where $DG(x)$ is the derivative or one of the derived numbers of $G(x)$ with respect to $g(x)$.

Mr. Maria† has made the important step in the proof of the lemma by showing that

$$G(x_2+0) - G(x_1-0) = \int_E DG(x)d\bar{g},$$

where E is the set $x_1 \leqq t \leqq x_2$. For the function $f_1(x)$, making use of Lemma 1, we have

$$\int_0^1 f_1(x)dG(x) = G(\alpha+0) - G(\alpha-0),$$

$$\begin{aligned}\int_0^1 f_1(x)DG(x)dg(x) &= \int_0^1 f_1(x)DG(x)d\bar{g} = \int_E DG(x)d\bar{g}\\ &= G(\alpha+0) - G(\alpha-0),\end{aligned}$$

where E is the point α. For $f_2(x)$ we have, if e is the open set $\alpha<x<\beta$,

$$\int_0^1 f_2(x)dG(x) = G(\beta-0) - G(\alpha+0),$$

$$\begin{aligned}\int_0^1 f_2(x)DG(x)dg(x) &= \int_0^1 f_2(x)DG(x)d\bar{g} = \int_e DG(x)d\bar{g}\\ &= G(\beta-0) - G(\alpha+0).\end{aligned}$$

From the above material the lemma readily follows.

* The same reasoning shows that $\int_0^x f(t)dh(t)$ is equal to $\int_0^x f(t)d\bar{h}$, for $0<x\leqq 1$, if $h(t)$ is continuous from the right except perhaps at $x=0$.

† Loc. cit., p. 430.

3. **Transformations.** Our first theorem is the following.

THEOREM 1. *If $G(x, y)$ is absolutely continuous $g(y)$ then equation* (1) *can be written in the form*

$$f(x) = m(x) + \lambda \int_0^1 K(x, y)f(y)dg(y), \tag{2}$$

where $K(x, y) = DG(x, y)$, the derivative being taken with respect to $g(y)$, a bounded monotone increasing function.

This is immediate from Lemma 2.

THEOREM 2. *If $m(x)$ and $K(x, y)$ are bounded, then the solution of* (1) *and* (2), *except for characteristic values of λ, can be written*

$$f(x) = m(x) + \lambda \int_0^1 \frac{D(x, y; \lambda)}{D(\lambda)} m(y)dg(y), \tag{3}$$

where

$$D(\lambda) = 1 - \lambda \int_0^1 K(s, s)dg(s) + \cdots ,$$

$$D(x, y; \lambda) = K(x, y) - \lambda \int_0^1 \begin{vmatrix} K(x, y) & K(x, s) \\ K(s, y) & K(s, s) \end{vmatrix} dg(s) + \cdots .$$

The proof follows along the same lines as in the ordinary case. We now state a corollary of Theorem 2 that represents most of the known results concerning solutions of equation (1).

COROLLARY.* *If $|G(x, y_2) - G(x, y_1)| \leqq |g(y_2) - g(y_1)|$, then, excepting characteristic values, equation* (1) *has* (3) *as a solution.*

Any result for the ordinary Fredholm equation carries a related result for equation (1). To see this, we assume without loss of generality that $g(y_1) < g(y_2)$ if $y_1 < y_2$, and apply to (2) the transformation†

$$\beta(s) = \limsup E_y(s \geqq g(y)), \qquad g(0) \leqq s \leqq g(1),$$

$$\begin{aligned} f(x) &= m(x) + \lambda \int_0^1 K(x, y)f(y)dg(y) \\ &= m(x) + \lambda \int_{g(0)}^{g(1)} K(x, \beta(s))f(\beta(s))ds. \end{aligned} \tag{4}$$

---

* This includes the case handled by W. C. Randels, *On Volterra-Stieltjes integral equations*, Duke Mathematical Journal, vol. 1 (1935), pp. 538–542.

† Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 6.

If we let ω be any of the possible solutions of

$$x = \beta(\omega), \qquad g(0) \leqq \omega \leqq g(1),$$

we may write (4) in the form

$$F(\omega) = M(\omega) + \lambda \int_{g(0)}^{g(1)} k(\omega, s)F(s)ds,$$

where $F(\omega) = f(\beta(\omega))$, $M(\omega) = m(\beta(\omega))$, $k(\omega, s) = K(\beta(\omega), \beta(s))$. We thus have our main result:

THEOREM 3. *When $G(x, y)$ is absolutely continuous $g(y)$ the Fredholm-Stieltjes integral equation* (1) *is reducible to an ordinary Fredholm integral equation.*

4. **Mixed linear equations.** The mixed equation*

$$f(x) = m(x) + \sum_{i=1}^{m} \lambda K^{(i)}(x)f(s_i) + \lambda \int_0^1 K(x, s)f(s)ds \tag{5}$$

can easily be put into the form

$$f(x) = m(x) + \lambda \int_0^1 R(x, s)f(s)dg(s).$$

Thus from Theorem 3 we see that equation (5) is reducible to a Fredholm integral equation.

DUKE UNIVERSITY

A THEOREM ON QUADRATIC FORMS†

WILLIAM T. REID

In this note the following result is proved:

THEOREM. *Suppose $A[x] \equiv a_{\alpha\beta}x_\alpha x_\beta$,‡ $B[x] \equiv b_{\alpha\beta}x_\alpha x_\beta$ are real quadratic forms in (x_α), $(\alpha = 1, \cdots, n)$, and that $A[x] > 0$ for all real $(x_\alpha) \neq (0_\alpha)$ satisfying $B[x] = 0$. Then there exists a real constant λ_0 such that $A[x] - \lambda_0 B[x]$ is a positive definite quadratic form.*

This theorem is of use in considering the Clebsch condition for multiple integrals in the calculus of variations. A. A. Albert§ has given

* W. A. Hurwitz, *Mixed linear integral equations of the first order*, Transactions of this Society, vol. 16 (1915), pp. 121–133.

† Presented to the Society, December 30, 1937.

‡ The tensor analysis summation convention is used throughout.

§ This Bulletin, vol. 44 (1938), pp. 250–253.