ON THE BOUNDARY CONDITION $\partial u/\partial n+au=0$ FOR HARMONIC FUNCTIONS*

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1. **Introduction.** In a recent paper (referred to below as I)† the author considered the nature of the "reflection" or analytic continuation of harmonic functions across a plane over which the boundary condition

$$\frac{\partial u}{\partial n} + au = 0, \qquad a = \text{const.}, \tag{1}$$

applies. In the following this investigation is continued. First, the case of a spherical (circular in two dimensions) boundary, over which the boundary condition (1) holds is considered. It is shown that singularities admit of a similar "reflection," as in the case of a plane in I; thus a point singularity at P_0 outside a sphere S over which (1) holds is reflected into a point singularity at P_1 (the spherical inverse of P_0 in S) and into a distribution of singularities along the radius vector from the center 0 to P_1.

Returning to plane boundaries, we apply boundary conditions of the form (1) over *two parallel* planes. A rather complicated "reflection" of singularities results, consisting of point singularities as well as of distributed line singularities. The point singularities are located at the periodic row of points obtained by reflecting the original singularity P_0 first in one plane, then in the other one; reflecting these images in the two planes; and so forth. The line singularities are distributed over the straight line through the above point singularities. The density of the distributed line singularities is an analytic function of the distance along the line bearing them between the point singularities, but changes abruptly from one analytic function to another one in crossing these points.

Some of the above features are believed to be typical of analytic continuations of a great variety of expansions in characteristic functions related to two point problems.

2. **Circular and spherical boundaries.** We shall consider the question of reflections of singularities of harmonic functions across spherical or circular boundaries corresponding to the condition (1).

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First we consider the two-dimensional harmonic case. Suppose that the function u satisfies

$$\frac{\partial u}{\partial r} - au = 0, \quad \text{for} \quad r = 1, \tag{2}$$

and is harmonic outside of this circle except near a point P_0 where it becomes infinite like $-\ln r_0$; u is also to vanish at infinity. The point P_0 is along the x axis at $x=h=e^H>1$. In terms of (two-dimensional) potential theory language, the singularity of u at P_0 is that of a "unit point charge."

The determination of u and its analytic continuation may be carried out by reducing it to the case considered in I by the application of the conformal transformation

$$z = e^Z, \qquad Z = X + iY, \qquad z = x + iy. \tag{3}$$

This straightens out the boundary $r=1$ by transforming it into the line $X=0$; the boundary condition (2) is preserved, while the unit point charge at $z=e^H$ is carried into unit point charges at

$$Z = \ln h = H + 2n\pi i. \tag{4}$$

Proceeding as in I, §5, we find that each one of the periodic array of point charges is "reflected" into a point charge and an exponential trail of negative charges. When we change back to the z plane, there results a positive image at $z=1/h$ and a distributed charge density

$$\rho(x) = -2ah^a x^{a-1}, \quad \text{for} \quad 0 < x < h. \tag{5}$$

In determining the latter, it is to be kept in mind that charge is preserved under conformal transformations, so that $\rho(x)$ is found from $\rho(x)dx=P(X)dX$, where $P(X)$ is the charge density along the X axis in the Z plane. From I, equation (23), $P(X)=-2ae^{a(h+x)}$, for $x<-h$.

As in I, the restriction I, (9), is pertinent. The total amount of charge then vanishes so that u vanishes at infinity like $O(|z|^{-1})$.

Proceeding to the three-dimensional case and a spherical boundary along which (2) holds, we can no longer utilize conformal transformations but proceed as follows.

In order to determine the function u which is harmonic except near $P_0:(x, y, z)=(h, 0, 0)$, $(h>1)$, where u becomes infinite like $1/r_0$ (the reciprocal distance from P_0) while along the unit sphere $r=1$ it satisfies the boundary condition (2), start with the expansion

$$u = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos\theta) + 1/r_0, \tag{6}$$

for u, $(r\geqq 1)$, and utilize the familiar expansions

$$(7)\qquad \frac{1}{r_0}=\left[\sum_{n=0}^{\infty}\frac{r^n}{h^{n+1}}P_n(\cos\theta)\quad\text{or}\quad\sum_{n=0}^{\infty}\frac{h^n}{r^{n+1}}P_n(\cos\theta)\right],$$

according as r is less than or greater than h, where θ is the angle with the x axis and P_n are the Legendre polynomials. If (2) is imposed,

$$(8)\qquad A_n=\frac{1}{h^{n+1}}-\frac{(1+2a)}{(n+a+1)h^{n+1}}\,.$$

The first part of A_n leads to a point charge of amount $1/h$ at P_1, the spherical inverse point of P_0; the latter leads to a distributed charge density $\rho(x)$ along the line from the center to P_1, provided $\rho(x)$ satisfies the moment equations

$$(9)\qquad \int_0^{1/h}\rho(x')(x')^n dx'=-\frac{(1+2a)}{(n+a+1)h^{n+1}},\qquad n=0,1,2,\cdots.$$

The solution of these equations is given by

$$(10)\qquad \rho(x)=\frac{(1+2a)x^a}{h^a}\,.$$

Magnification can be used for treating the case where (2) holds along a sphere of non-unit radius, while singularities inside the circle or sphere can be reduced to outer singularities by means of inversion.

The two-dimensional case can also be treated in a similar fashion with Fourier series replacing the series of Legendre polynomials.

3. **Boundary conditions along two parallel planes.** We consider in this section continuations of harmonic functions across two parallel planes $x=0$, $x=C$, corresponding to boundary conditions of the form (1) on each plane, though for simplicity we suppose that (1) applies along $x=0$. Then

$$(11)\qquad \frac{\partial u}{\partial x}-au=0,\qquad\text{for}\quad x=0,$$

$$(12)\qquad \frac{\partial u}{\partial x}=0,\qquad\text{for}\quad x=C.$$

We shall investigate the reflections of a point charge singularity located at P_0: $(h, 0, 0)$ between the planes $x=0$, $x=C$, $(0<h<C)$.

One way of obtaining the singularities of the analytic continuation is to apply the familiar method of successive reflections, attending alternately first to $x=0$, then to $x=C$. There result point charges along the x axis at

$$x = \pm h \pm 2nC, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{13}$$

as well as proper continuous charge distributions which change abruptly to different analytic functions on crossing the points (13). This process of reflection gets rather irksome after several stages.

We shall first follow these reflections by means of the operational treatment employed in I, §8. If $S_0(x-h)$ is the original charge at $x=h$, reflection across $x=C$ leads to the new charge $S_0(x-2C+h)$. As shown in I, §8, analytic continuation across $x=0$ results in multiplication by the operator $(p+a)/(p-a)$, in addition to replacement of the argument by its negative. This introduces the charges

$$\frac{p+a}{p-a}\left[S_0(x+h) + S_0(x+2C-h)\right]. \tag{14}$$

Further reflection in $x=C$ replaces x by $2C-x$, changes $f(p)$ to $-f(-p)$, and results in the charges

$$\frac{a-p}{a+p}\left[S_0(x-2C-h) + S_0(x-4C+h)\right]. \tag{15}$$

Following this by a reflection in $x=0$ we would apparently have

$$\begin{aligned}\frac{p+a}{p-a}\,\frac{a-p}{a+p}&\left[S_0(x+2C+h) + S_0(x+4C-h)\right]\\ &= -\left[S_0(x+2C+h) + S_0(x+4C-h)\right].\end{aligned} \tag{16}$$

This, however, may be shown to be incorrect. The explanation of the difficulties thus encountered by the operational treatment will appear presently.

We shall give a Bessel-Fourier integral representation for u. Utilizing I, equation (16), we put

$$u = \int_0^\infty J_0(\lambda\rho)\left[e^{-\lambda|x-h|} + f(\lambda)e^{\lambda x} + g(\lambda)e^{-\lambda x}\right]d\lambda, \tag{17}$$

where $f(\lambda)$, $g(\lambda)$ are to be determined. Imposing the conditions (11), (12) on the integrand and solving, say for $f(x)$, we obtain

$$f(\lambda) = \frac{e^{+\lambda h}\dfrac{\lambda+u}{\lambda-a} + e^{-\lambda h}}{e^{2\lambda C}\dfrac{\lambda+a}{\lambda-a} - 1}\cdot \tag{18}$$

Expanding $f(\lambda)$ in powers of $e^{-2\lambda C}(\lambda-a)/(\lambda+a)$ one obtains

$$(19)\quad f(\lambda) = e^{\lambda(h-2C)}\sum_{n=0}^{\infty}\left(\frac{\lambda - a}{\lambda + a}\right)^{n} e^{-2\lambda nC} + e^{-\lambda h}\sum_{n=0}^{\infty}\left(\frac{\lambda - a}{\lambda + a}\right)^{n+1} e^{-2\lambda nC},$$

and the substitution of (19) in the middle term of (17) results in the potential function

$$(20)\quad \begin{aligned} &\sum_{n=0}^{\infty}\int_{0}^{\infty}\left(\frac{\lambda - a}{\lambda + a}\right)^{n} e^{\lambda[x+h-2(n+1)C]}J_0(\lambda\rho)d\lambda \\ &\quad + \sum_{n=0}^{\infty}\int_{0}^{\infty}\left(\frac{\lambda - a}{\lambda + a}\right)^{n+1} e^{\lambda(x-h-2nC)}J_0(\lambda\rho)d\lambda. \end{aligned}$$

To interpret (20) as the potential of charges, note that from I, equation (16), it follows that a charge distribution $c(x)$ over the positive x axis gives rise to the potential

$$(21)\quad \int_{0}^{\infty} f(\lambda)e^{\lambda x}J_0(\lambda\rho)d\lambda, \qquad x < 0,$$

where

$$(22)\quad f(\lambda) = \int_{0}^{\infty} c(x)e^{-\lambda x}dx.$$

Thus, if $f(\lambda)$ is given (say by means of (18) or (19)), one obtains for $c(x)$ a Laplace-Carson integral equation. The solution of the latter can be effected by means of a Bromwich integral; in operational notation the solution appears in the form

$$(23)\quad c(x) = f(p)S_0(x).$$

The explicit charge corresponding to the displayed term in the first series in (20) is thus

$$(24)\quad \left(\frac{p - a}{p + a}\right)^{n} e^{p[h-2(n+1)C]}S_0(x).$$

The effect of the exponential is to translate the distribution a distance $2(n+1)C-h$ in the direction of positive x; there remains to interpret

$$(25)\quad \left(\frac{p - a}{p + a}\right)^{n} S_0(x) = \left(1 - \frac{2a}{p + a}\right)^{n} S_0(x).$$

This is done by expanding by the binomial theorem and replacing $(p+a)^{-k}S_0(x)$ by $x^{k-1}e^{-ax}/(k-1)!$, for $x>0$, and by 0 for $x<0$. Of course, the first term in the expansion of the right-hand side of (25) leads to a unit point charge at $x=0$. It follows that (24) represents a point charge at $x=2(n+1)C-h$ and a continuous charge beyond this point of charge density $P_n(x)e^{-\alpha x}$, where $P_n(x)$ is a polynomial of degree n.

As regards the function $g(\lambda)$ and the last integral in (17), a similar treatment is possible resulting in an interpretation in terms of charges along the negative x axis. The latter might be described essentially as the positive image of the charges for $x>C$ in the plane $x=C$. However, in place of (22) an integral equation of the form

$$g(\lambda) = \int_{-\infty}^{0} c_1(x)e^{\lambda x}dx \tag{26}$$

now obtains for the charge density $c_1(x)$ along $x<0$. A Bromwich integral will not yield the solution of this equation unless one changes the independent variable to $-x$ or properly changes the path of integration.

An operational solution of (26) (similar to the solution (23) of (22)) could therefore be given only by using a non-orthodox interpretation of the operator $g(p)$. Essentially the same situation appeared in I, where, in equation (42), $1/(p-\alpha)S_0(x)$ was interpreted as $-e^{\alpha x}H_0(-x)$ rather than $e^{\alpha x}H_0(x)$, as is usually done in operational calculus* (see also I, footnote, p. 884). This also accounts for the difficulties encountered in connection with (16) and the attempted operational treatment of the successive reflections (two distinct (and mutually inconsistent) interpretations of the operators $1/(p-a)$, $1/(p+a)$ occurring in (15), (16) have to be employed).†

A very similar treatment can be given for the case wherein general linear conditions of the form (11) apply along both $x=0$ and $x=C$.

We believe that some of the features of the above analytic continuation are characteristic of general differential systems with two point boundary conditions. It is planned to touch upon this phase in the future.

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* Both interpretations yield the Green's function of the differential equation $(d/dx-\alpha)u=0$ with discontinuity at $x=0$, the one relative to the boundary condition $u(+\infty)=0$, the other relative to $u(-\infty)=0$.

† Roughly speaking, the operational method fails here because of its catering to functions which vanish for sufficiently negative x. It can handle the charges for $x>0$. Properly modified (as in I, §8) it can handle the charges for $x<0$. It is not adapted, however, to treating both sets of charges and exhibiting their relations.