

ON THE PEANO CURVE OF LEBESGUE*

I. J. SCHOENBERG

Let $f(t)$ be the *even* continuous function of period two which is defined in the interval $(0, 1)$ as follows: $f(t) = 0$ in $(0, 1/3)$, $f(t) = 1$ in $(2/3, 1)$, and $f(t)$ is linear in $(1/3, 2/3)$. Our curve is defined by the parametric equations

$$(1) \quad \begin{aligned} x(t) &= \frac{1}{2} f(t) + \frac{1}{2^2} f(3^2 t) + \frac{1}{2^3} f(3^4 t) + \cdots, \\ y(t) &= \frac{1}{2} f(3t) + \frac{1}{2^2} f(3^3 t) + \frac{1}{2^3} f(3^5 t) + \cdots, \quad 0 \leq t \leq 1. \end{aligned}$$

The inequalities $0 \leq f(t) \leq 1$ imply $0 \leq x(t) \leq 1$, $0 \leq y(t) \leq 1$, as well as the uniform convergence of both series (1), and hence imply the continuity of $x(t)$, $y(t)$. All there remains to show is that our curve will pass through an arbitrarily given point

$$(2) \quad x_0 = \frac{a_0}{2} + \frac{a_2}{2^2} + \frac{a_4}{2^3} + \cdots, \quad y_0 = \frac{a_1}{2} + \frac{a_3}{2^2} + \frac{a_5}{2^3} + \cdots, \quad a_\nu = 0, 1,$$

of the square $0 \leq x, y \leq 1$, whose coordinates are given by their binary expansions. Indeed, let

$$(3) \quad t_0 = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \cdots + \frac{2a_{k-1}}{3^k} + \frac{2a_k}{3^{k+1}} + \cdots.$$

If $a_0 = 0$, we have $0 \leq t_0 \leq 2/3^2 + 2/3^3 + \cdots = 1/3$, hence $f(t_0) = 0$; if $a_0 = 1$, we have $2/3 \leq t_0 \leq 2/3 + 1/3 = 1$, hence $f(t_0) = 1$. In either case $f(t_0) = a_0$. Similarly $3^k t_0 = \text{even integer} + 2a_k/3 + 2a_{k+1}/3^2 + \cdots$ shows that

$$(4) \quad f(3^k t_0) = a_k, \quad k = 0, 1, 2, \cdots.$$

Now (1), (2), and (4) imply $x(t_0) = x_0$, $y(t_0) = y_0$.†

COLBY COLLEGE

* Presented to the Society, April 16, 1938.

† Lebesgue, *Leçons sur l'Intégration*, Paris, 1928, pp. 44-45, defines the functions $x(t)$, $y(t)$, first on Cantor's ternary set T , of points t_0 of the form (3), by means of the equations (2). Having proved their continuity on T , Lebesgue extends their definition throughout $(0, 1)$ by linear interpolation over each one of the denumerable set of open intervals of which the set complementary to T is built up. It should be remarked that our curve (1) coincides with Lebesgue's curve within Cantor's set T but not on the complement of T .