SHORTER NOTICES

Leçons sur les Principes Topologiques de la Théorie des Fonctions Analytiques. By S. Stöilow. Paris, Gauthier-Villars, 1938. 10+148 pp.

This volume is another of the well known series, *Collection de Monographies sur la Théorie des Fonctions*, edited by É. Borel. Its purpose is to study the purely topological aspects of the theory of Riemann surfaces and of analytic functions, and to derive some standard theorems and generalizations from this point of view. The key theorem here, that an "interior transformation" is topologically equivalent to an analytic function, was first proved by Stöilow in 1928, in the Annales de l'École Normale, (3), vol. 45, p. 367.

Of course, standard parts of classical function theory are partly topological in nature; we may mention the theorem of Stokes (on which Cauchy's integral theorem is based), the monodromy theorem, and the fundamental theorem of algebra. However, it is not with this side of the subject that Stöilow is concerned, for these theorems cannot be put in purely topological form. The author assumes that the reader is acquainted with classical function theory, including its topological aspects, centering around the Jordan curve theorem.

The first chapter is an introduction to the general theory of topological spaces and of manifolds (particularly 2-dimensional manifolds). Different postulate systems are given, and such topics as the properties of open and closed sets, neighborhoods, compact spaces, and connected sets are studied. It must be said that several definitions and theorems (such as a definition of the term "totalement discontinue," and the theorem that a $(1-1)$ continuous transformation of a compact space is a homeomorphism) are not given, though they are used in later chapters. The second half of the chapter is devoted to the theorem of Brouwer on the invariance of regions. The n-dimensional case is given, using the Sperner proof of the Lebesgue lemma. The author remarks that the 2-dimensional case, which is all that is needed in the book, may be proved much more simply. The average reader will wish that he had given such a proof.

Riemann surfaces are defined, and their relations to analytic functions are given, in the second chapter. Unlike Weyl, Stöilow defines a Riemann surface as being a system composed of a surface V, together with a mapping f of this surface on the (extended) complex plane V_0, certain conditions being satisfied. These conditions are that the interiors of a finite or denumerable set $\delta_1, \delta_2, \cdots$ of closed regions cover V and that, for each i, f be topologically equivalent to $w=z^{n_i}$ for some n_i, in δ_i. That is, there are topological mappings h and h_0 of δ_i and $f(\delta_i)$ into the unit circle in the complex plane such that

$$(1) \qquad h_0(f(h^{-1}(z))) = z^{n_i}.$$

(In the definition of Weyl, the function f is not given, but it is assumed that there is an analytic metric given in V; his assumption of the triangulability of V was shown to be unnecessary by Radó.) It is easily seen that the Riemann surface V corresponding to an analytic function f satisfies these conditions. The converse is also proved here, using the method of Weyl, Courant, and Fatou.

The third and fourth chapters are devoted to a study of 2-dimensional manifolds in general, determining what ones can be Riemann surfaces (that is, can be the part V of the pair V, f), and classifying these. To make V a Riemann surface, one must find a suitable mapping of V on the complex plane. This is possible if V can be covered by a finite or denumerable number of neighborhoods homeomorphic to a region in the plane, and if V is orientable. The classification follows the standard procedure, due to Jordan for closed surfaces, and to Kerékjártó for open surfaces.

In Chapter 5, the topological characterization of analytic functions is given. A transformation f of the space X into the space X_0 is called *equivalent* to the transformation f' of X' into X_0' if there are topological transformations h of X into X' and h_0 of X_0 into X_0' such that

$$f'(p) = h_0(f(h^{-1}(p))).$$

Two properties of mappings which are invariant under this equivalence are the following: The image of any open set is an open set, and no closed connected set containing more than one point goes into a single point. Mappings with these properties are called *interior*. (This term, or *inner*, is now commonly used by topologists to refer to mappings satisfying the first property.) The fundamental theorem is that any analytic function, as a mapping of its Riemann surface on the complex plane, is interior, and conversely, any interior mapping of a surface on the complex plane is equivalent to an analytic function for which the given surface can be taken as its Riemann surface. The essential step in the proof is to show that an interior transformation behaves locally like z^n for some n, as in (1).

The last chapter gives some applications of preceding methods and results, especially to properties of transformations of one Riemann surface into another. The formula of Hurwitz, relating the genus and number of boundaries of each surface to the degree of the transformation and the total amount of branching, is given and generalized. Asymptotic and limiting values of an analytic function are discussed.

The book should prove of real interest to anyone wishing to study deeply into the underlying topological properties of Riemann surfaces and analytic functions. On the whole, the exposition is quite clear, though here and there one finds slight errors and omissions of important details.

HASSLER WHITNEY

Opérations Infinitésimales Linéaires. By Vito Volterra and Bohuslav Hostinsky. Paris' Gauthiers-Villars, 1938. 7+238 pp.

The infinitesimal calculus of linear operators was invented by Volterra in 1887, and it is with unusual interest that one opens a volume written fifty years later on this important subject, when one discovers that he is a coauthor.

Consider a linear operator X which is a function $X(t)$ of the time t. If multiplication is taken as the fundamental operation,* then analogy with ordinary functions suggests letting the quotient $X(t+\Delta t)X^{-1}(t)$ measure the "change" in X during the interval from t to $t+\Delta t$, and

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[X(t+\Delta t)X^{-1}(t)\right] \tag{1}$$

measure the "rate of change," or "derivative" (more properly, right-derivative) of $X(t)$. It is natural to regard this derivative as a sort of "infinitesimal linear operator," whence the title of the book.

* If addition is taken as the fundamental operation, one gets the (commutative) infinitesimal calculus of vectors, which was discussed by H. Grassmann in 1862, in his *Ausdehnungslehre*, part 2, chaps. 2–4.