

**THE EQUIVALENCE OF SEQUENCE INTEGRALS AND
NON-ABSOLUTELY CONVERGENT INTEGRALS***

R. L. JEFFERY

This note completes and extends some results previously obtained. † Let the function $f(x)$ be measurable, and finite almost everywhere on (a, b) . Let $s_n(x)$ be a sequence of summable functions such that $s_n = f$ on a set E_n , $s_n = 0$ elsewhere, $E_n \supset E_{n-1}$, and $mE_n \rightarrow b - a$. If $\int_a^x s_n dx$ tends to a continuous function $\phi(x)$, then f is, by definition, *totally integrable in the sequence sense* to $\phi(x) = TS(f, a, x)$. It has been shown that if $f(x)$ is integrable in the generalized Denjoy sense to $F(x) = \int_a^x f(x) dx$, then there exists $TS(f, a, x) = F(x)$. ‡ Such a function $TS(f, a, x)$ is generalized absolutely continuous (ACG), § since $F(x)$ is (ACG). A function $TS(f, a, x)$ was constructed || which was not (ACG) and consequently not equal to $F(x)$. This raised the question as to whether or not the property of being (ACG) was sufficient to insure that $TS(f, a, x) = F(x)$. In the present note this question is answered in the negative, and necessary and sufficient conditions are determined for the relation $TS(f, a, x) = F(x)$.

We first construct a function $f(x)$ which is not summable, but which is integrable in a non-absolutely convergent sense, and then construct $TS(f, a, x)$ which is (ACG) and not equal to $F(x) = \int_a^x f dx$. Let G be a Cantor set on (a, b) with $mG > 0$, and let (α_i, β_i) be the intervals complementary to G . On (α_i, β_i) construct f_i such that $\int_{\alpha_i}^{\beta_i} f_i dx$ exists as a non-absolutely convergent integral with β_i the single point of non-summability of f_i , with $\int_{\alpha_i}^{\beta_i} f_i dx = 0$, and with $|\int_{\alpha_i}^x f_i dx| < \beta_i - \alpha_i$ for x on (α_i, β_i) . Let $f(x) = f_i(x)$ on (α_i, β_i) , and $f(x) = 0$ elsewhere. Then $F(x) = \int_a^x f dx$ exists as a non-absolutely convergent integral, and $F(x) = 0$ for x a point of G . Consider the set of intervals (α_i, β_i) ordered in any way. Then take the first n intervals of this set and order them from left to right into the set $(\alpha'_1, \beta'_1), \dots, (\alpha'_n, \beta'_n)$. To the right of each interval (α'_j, β'_j) there is an interval $\lambda_{ni} = (\beta'_j, \alpha'_{j+1})$, where i is the subscript that (α_i, β_i) has in the original ordering

* Presented to the Society, January 1, 1936.

† Transactions of this Society, vol. 41 (1935), pp. 171-192. In what follows this paper will be referred to as T.

‡ T, p. 186, Theorem 6.

§ Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 152, §9.

|| T, pp. 189-191.

(α_i, β_i) . On (α'_j, β'_j) fix a set E_{ni} such that if $s_{ni}=f$ on E_{ni} and $s_{ni}=0$ elsewhere, then

$$\left| \int_{\alpha'_j}^{\beta'_j} s_{ni} dx - m\lambda_{ni} \right| < \frac{1}{n^2}.$$

If $s_n = s_{ni}$ on (α'_j, β'_j) and $s_n = 0$ elsewhere, then

$$\left| \int_a^b s_n dx - \sum m\lambda_{ni} \right| < \frac{1}{n}.$$

As n increases it is possible to choose the sets E_{ni} in such a way that $E_{(n+1)i} \supset E_{ni}$ and $mE_{ni} \rightarrow \beta_i - \alpha_i$. If this is done, it is then easily verified that

$$\lim_{n \rightarrow \infty} \int_a^x s_n dx = F(x) + mG(a, x) = TS(f, a, x),$$

where $G(a, x)$ is the part of G on (a, x) . The function $F(x)$ is (ACG), and the function $mG(a, x)$ is (AC). Hence $TS(f, a, x)$ is (ACG). Furthermore, since $mG > 0$, we have $TS(f, a, x) \neq F(x)$.

The foregoing considerations lead us to seek necessary and sufficient conditions that $TS(f, a, x) = F(x)$. Associated with the function $\phi(x) = TS(f, a, x)$ is a function of sets $\phi(G) = \lim \int_G s_n dx$, provided this limit exists, where G is a measurable set on (a, b) and s_n is the sequence involved in the definition of $TS(f, a, x)$. If G is an interval (α, β) , then $\phi(\alpha, \beta) = TS(f, \alpha, \beta)$. The function of sets $\phi(G)$ is *completely additive* if, for every set of disjoint sets G_1, G_2, \dots , we have the relation $\phi(\sum G_i) = \sum \phi(G_i)$. The function $\phi(G)$, associated with the function $f(x)$ defined above, is not completely additive. For if (α_i, β_i) is the set of open intervals complementary to the set E , then $\phi[\sum(\alpha_i, \beta_i)] = mE$, while $\sum \phi(\alpha_i, \beta_i) = 0$. We prove the following theorem:

THEOREM 1. *If $f(x)$ is integrable in the generalized Denjoy sense to $F(x)$, and if $\phi(x) = TS(f, a, x)$ is such that $\phi(G)$ is completely additive, then $\phi(x) = F(x)$.*

Let E_1 be the points of non-summability of f over (a, b) , (α_i, β_i) the intervals on (a, b) complementary to E_1 , and (α'_i, β'_i) an interval with $\alpha_i < \alpha'_i < \beta'_i < \beta_i$. The function f is summable on (α'_i, β'_i) , and

$$\phi(\beta'_i) - \phi(\alpha'_i) = \int_{\alpha'_i}^{\beta'_i} f dx = F(\beta'_i) - F(\alpha'_i).$$

It then follows from the continuity of F and ϕ that $\phi(\beta_i) - \phi(\alpha_i) = F(\beta_i) - F(\alpha_i)$. Let E_2 be the points of non-summability of f over E_1

together with the points of E_1 at which $\sum |F(\beta_i) - F(\alpha_i)|$ diverges, let (α_j, β_j) be the intervals on (a, b) complementary to E_2 , and let (α'_j, β'_j) be an interval with $\alpha_j < \alpha'_j < \beta'_j < \beta_j$. Then

$$\phi(\beta'_j) - \phi(\alpha'_j) = \lim_{n \rightarrow \infty} \int_{\alpha'_j}^{\beta'_j} s_n dx = \lim_{n \rightarrow \infty} \int_{\Sigma(\alpha_i, \beta_i)} s_n dx + \lim_{n \rightarrow \infty} \int_e s_n dx,$$

where e is the part of E_1 on (α'_j, β'_j) , and $\Sigma(\alpha_i, \beta_i)$ is the part of the set (α_i, β_i) on (α'_j, β'_j) . The second limit on the right exists for the reason that f is summable over e . Consequently the first limit on the right exists. Then, since $\phi(G)$ is completely additive,

$$\begin{aligned} \phi(\beta'_j) - \phi(\alpha'_j) &= \sum_l \lim_{n \rightarrow \infty} \int_{\alpha_l}^{\beta_l} s_n dx + \int_e f dx \\ &= \sum \{ \phi(\beta_l) - \phi(\alpha_l) \} + \int_e f dx \\ &= \sum \{ F(\beta_l) - F(\alpha_l) \} + \int_e f dx \\ &= F(\beta'_j) - F(\alpha'_j). \end{aligned}$$

Again the continuity of F and ϕ gives $\phi(\beta_j) - \phi(\alpha_j) = F(\beta_j) - F(\alpha_j)$. This process can be continued by means of finite and transfinite induction to give $\phi(x) = F(x)$ for x on (a, b) .

The complete additivity of $\phi(G)$ is not a necessary condition that $\phi(x) = TS(f, a, x) = F(x)$. Let $x_0 = 0 < x_1 < x_2 < \dots$ be a sequence of values of x on $(0, 1)$ with x_n tending to unity. Let f be so defined on (x_{n-1}, x_n) that the integral of f over this interval exists as a non-absolutely convergent integral with x_n the single point of non-summability. Furthermore, let f be such that if $F(x) = \int_a^x f dx$, then $F(x_n) - F(x_{n-1}) = (-1)^{n-1}n$. There then exists $TS(f, a, x)^*$ such that

$$\phi(x) = TS(f, a, x) = \lim_{n \rightarrow \infty} \int_0^x s_n dx = F(x).$$

The set E of points of non-summability of f is the set x_0, x_1, \dots , and unity. Let (α_i, β_i) be the intervals complementary to E . We have

$$\begin{aligned} F(1) - F(0) &= \lim_{n \rightarrow \infty} \int_0^1 s_n dx = \lim_{n \rightarrow \infty} \int_E s_n dx + \lim_{n \rightarrow \infty} \int_{CE} s_n dx \\ &= \lim_{n \rightarrow \infty} \int_{\Sigma(\alpha_i, \beta_i)} s_n dx = \phi[\sum (\alpha_i, \beta_i)]. \end{aligned}$$

* T, p. 186, Theorem 6.

But the value of $\sum \phi(\alpha_i, \beta_i)$ depends on the order in which the intervals (α_i, β_i) are taken.

THEOREM 2. *Let $F(x) = \int_a^x f dx$, where the integration is in the generalized Denjoy sense. Let $\phi(x) = TS(f, a, x)$. A necessary and sufficient condition that $\phi(x) = F(x)$ is that if (l, m) is an interval on (a, b) containing a closed set e over which f is summable, if (α_i, β_i) are the intervals on (l, m) contiguous to e , and if $\sum |\phi(\beta_i) - \phi(\alpha_i)|$ converges, then $\phi[\sum(\alpha_i, \beta_i)] = \sum \phi(\alpha_i, \beta_i)$.*

The proof of the sufficiency of the conditions follows the same lines as the proof of Theorem 1. To show that the conditions are necessary let $\phi(x) = TS(f, a, x) = F(x)$. Then

$$\begin{aligned} \phi(m) - \phi(l) &= \lim_{n \rightarrow \infty} \int_{\sum(\alpha_i, \beta_i)} s_n dx + \lim_{n \rightarrow \infty} \int_e s_n dx \\ &= \phi[\sum(\alpha_i, \beta_i)] + \int_e f dx. \end{aligned}$$

From this, and the fact that $\phi(x) = F(x)$, it follows that $\phi[\sum(\alpha_i, \beta_i)] = \sum \{F(\beta_i) - F(\alpha_i)\} = \sum \{\phi(\beta_i) - \phi(\alpha_i)\}$.

In the foregoing f is summable over e . Suppose that f is not summable over e , but that $\int_e f dx$ exists as a non-absolutely convergent integral, and suppose that $\sum |\phi(\beta_i) - \phi(\alpha_i)|$ converges. Is it necessary that $\phi[\sum(\alpha_i, \beta_i)] = \sum \phi(\alpha_i, \beta_i)$? We answer this question in the negative. Returning to the first example above, we bisect the intervals (α_i, β_i) , getting the intervals (α_i, a_i) , (a_i, β_i) . On each of these intervals we define f_i in the way that f_i was defined on the original interval (α_i, β_i) . In particular, a_i will be the single point of non-summability of f_i on (α_i, a_i) , and β_i will be the single point of non-summability of f_i on (a_i, β_i) . Let $f = f_i$ on (α_i, a_i) and on (a_i, β_i) , and let $f = 0$ elsewhere. Going to the interval (α'_j, β'_j) we determine sets E''_{ni} on (α'_j, a_i) and E''_{ni} on (a_i, β'_j) in such a way that if $s'_n = f$ on E''_{ni} , and $s_n = 0$ elsewhere, then $\int_a^x s_n dx \rightarrow mG(a, x)$; if $s''_n = f$ on E''_{ni} and $s''_n = 0$ elsewhere, then $\int_a^x s_n dx \rightarrow -mG(a, x)$. If further $s_n = s'_n + s''_n$, then

$$\phi(x) = \lim_{n \rightarrow \infty} \int_a^x s_n dx = TS(f, a, x) = \int_a^x f dx = F(x).$$

Let e be the closed set complementary to the set of open intervals (α_i, a_i) . Then $\int_e f dx$ exists as a non-absolutely convergent integral. But $\phi[\sum(\alpha_i, a_i)] = mG$, and $\sum \phi(\alpha_i, a_i) = 0$. It thus appears that in so far as it is a question of complete additivity, the conditions of Theorem 2 are the best possible.

So far it has been assumed that $f(x)$ is integrable in a non-absolutely convergent sense. We now prove the following theorem:

THEOREM 3. *Let the function $f(x)$ be measurable on (a, b) , and let $TS(f, a, x)$ exist. If E is any closed set on (a, b) , let an interval (l, m) containing a part e of E exist such that f is summable over e , $\sum |\phi(\alpha_i, \beta_i)|$ converges, and $\sum \phi(\alpha_i, \beta_i) = \phi[\sum(\alpha_i, \beta_i)]$, where (α_i, β_i) are the intervals complementary to e on (l, m) . Then f is integrable in the generalized Denjoy sense, and $\int_a^x f dx = TS(f, a, x)$.*

If the closed set E of the theorem is the interval (a, b) , then the part e of E on (l, m) is all of (l, m) , and it follows that the points E_1 of non-summability of f over (a, b) are non-dense on (a, b) . Let (α, β) be an interval complementary to E_1 . The function f is then summable over any interval interior to (α, β) , and as a consequence of this $\phi' = f$ almost everywhere on (α, β) . The set E_1 is closed. By the conditions of the theorem there is an interval (l, m) containing a part e of E_1 with f summable over e , $\sum |\phi(\alpha_i, \beta_i)|$ convergent, and $\sum \phi(\alpha_i, \beta_i) = \phi[\sum(\alpha_i, \beta_i)]$. For x on this interval (l, m) , we have

$$\begin{aligned} \phi(x) - \phi(l) &= \lim_{n \rightarrow \infty} \int_l^x s_n dx \\ &= \lim_{n \rightarrow \infty} \int_{e(l, x)} s_n dx + \lim_{n \rightarrow \infty} \int_{\Sigma(l, x)(\alpha_i, \beta_i)} s_n dx \\ &= \int_e f dx + \sum_{(l, x)} \phi(\alpha_i, \beta_i) + \phi(\alpha_k, x), \end{aligned}$$

where the second term on the right represents the whole intervals of the set (α_i, β_i) on (l, x) , and the third term is absent unless x is an interior point of an interval (α_k, β_k) of the set (α_i, β_i) . Set $\psi(x) = \phi_1(x) + \phi_2(x)$, where

$$\phi_1 = \int_{e(l, x)} f dx, \quad \phi_2 = \sum_{(l, x)} \phi(\alpha_i, \beta_i).$$

The function ψ is constant on (α_i, β_i) and equal to ϕ at points of e . Furthermore, $\phi'_1 = f$ at almost all of e , and $\phi'_2 = 0$ at almost all of e .* Consequently $\psi' = f$ almost everywhere on e , and since $\psi = \phi$ at points of e , it follows that ϕ has an approximate derivative equal to f at almost all points of e . If now E_2 denotes the points of E_1 which are points of non-summability of f over E , or the points of E_1 at

* Denjoy, *Journal de Mathématique*, (7), vol. 1, p. 158 ff.

which $\sum |\phi(\alpha_i, \beta_i)|$ either diverges or converges with $\sum \phi(\alpha_i, \beta_i) \neq \phi[\sum(\alpha_i, \beta_i)]$, then the conditions of the theorem and the above reasoning allow us to conclude that the set E_2 is non-dense on E_1 , and that if (α, β) is an interval of the set complementary to E_2 , then ϕ has an approximate derivative equal to f at almost all points of this interval. This process can be continued by finite and transfinite induction to show that ϕ has an approximate derivative equal to f almost everywhere on (a, b) .

It will next be shown that $\phi(x) = TS(f, a, x)$ is (ACG). It is sufficient to show that for every closed set E there exists an interval containing a part e of E over which ϕ is absolutely continuous.* Let E be any closed set on (a, b) , and (l, m) an interval containing a part e of E over which f is summable and for which $\sum |\phi(\alpha_i, \beta_i)|$ converges with $\sum \phi(\alpha_i, \beta_i) = \phi[\sum(\alpha_i, \beta_i)]$. Let (a', b') be an interval on (l, m) with a', b' points of e . Then

$$|\phi(b') - \phi(a')| = \left| \int_{e(a', b')} f dx + \sum_{(a', b')} \phi(\alpha_i, \beta_i) \right|.$$

Since $\sum \phi(\alpha_i, \beta_i)$ converges, it is clear that the right-hand side of this equation can be made arbitrarily small by taking $b' - a'$ sufficiently small. Similarly, $\sum |\phi(b'_k) - \phi(a'_k)|$ is arbitrarily small if (a'_k, b'_k) is a finite or denumerably infinite set of intervals of (l, m) with a'_k, b'_k points of e and $\sum (b'_k - a'_k)$ sufficiently small. But this means that ϕ is (AC) on e and consequently (ACG) on (a, b) .

We now have $\phi(x)$ (ACG) on (a, b) and the approximate derivative of ϕ equal to f almost everywhere. This allows us to conclude that f is integrable in the generalized Denjoy sense,† and that

$$\phi(x) = \int_a^x f dx = TS(f, a, x).$$

THE UNIVERSITY OF WISCONSIN

* Saks, loc. cit., p. 165.

† Saks, loc. cit., p. 197, §2.