

ON FOURIER SERIES WITH RESTRICTED COEFFICIENTS*

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1. **Introduction.** Consider a real-valued function $f(x)$, periodic with period 2π and Lebesgue integrable. Let

$$(1.1) \quad f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

be its Fourier series, and let $s_0 = a_0/2$,

$$(1.2) \quad s_n(f; x) = s_n = \frac{a_0}{2} + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x),$$

$n = 1, 2, 3, \dots,$

be its partial sums. We shall mainly restrict ourselves to series satisfying the conditions

$$(1.3) \quad na_n \geq -p, \quad nb_n \geq -p, \quad \text{for } n = 1, 2, 3, \dots,$$

where $p \geq 0$. We shall in particular consider the following problem:

Suppose

$$(1.4) \quad -\mu \leq f(x) \leq \mu \quad \text{in } -\pi < x < \pi;$$

then what is the best upper bound $C_n(\mu, p)$ for the partial sums $|s_n(f; x)| \leq C_n(\mu, p)$, ($n \geq 1$), under the assumption (1.3)?

It is known that the sequence $\{C_n\}$ is bounded (cf. Szász [5], [6]);† hence l.u.b. $C_n(\mu, p) = C(\mu, p)$ is finite. For $p=0$ the author [9] proved recently that

$$(1.5) \quad C(\mu, 0) < (2 + 4/\pi)\mu < 3.3\mu;$$

for $p > 0$ the sharpest estimates so far were given by Fekete [2] using a device of Paley and Fejér [1]. Fekete proved that

$$(1.6) \quad C(\mu, p) < 5\mu + 6p,$$

and also that

$$(1.7) \quad C(\mu, p) < 5\mu + 8(\mu p)^{1/2}.$$

* Presented to the Society, April 9, 1938.

† See the list of references at the end of this paper.

Note that for any $\rho > 0$, we have $s_n(\rho f; x) = \rho s_n(f; x)$, which gives the relation $\rho C_n(\mu, \rho) = C_n(\rho\mu, \rho)$, ($\rho \geq 0, \mu > 0$), and in particular

$$(1.8) \quad C_n(\mu, \rho) = \mu C_n(1, \rho/\mu).$$

Hence in the discussion we may put $\mu = 1$; that is, $|f(x)| \leq 1$. Using a similar tool as in [9] we shall improve upon the known results. We shall get sharper estimates for the k th partial sum assuming (1.3) only for $n = 1, 2, \dots, 2k - 1$.

2. Certain identities and inequalities. Given an infinite series $\sum_0^\infty u_\nu$, let $\sum_0^n u_\nu = U_n, \sum_0^{n-1} U_\nu = U_n^{(1)} = \sum_0^n (n - \nu)u_\nu, (n = 0, 1, 2, \dots)$; we then have

$$(2.1) \quad \begin{aligned} \frac{1}{2n} U_{2n}^{(1)} - \frac{1}{n} U_n^{(1)} &= \frac{1}{2n} \left\{ \sum_0^{2n} (2n - \nu)u_\nu - 2 \sum_0^n (n - \nu)u_\nu \right\} \\ &= \frac{1}{2n} \left\{ \sum_0^n \nu u_\nu + \sum_{n+1}^{2n} (2n - \nu)u_\nu \right\}, \end{aligned}$$

for $n = 1, 2, \dots$, and $U_n^{(1)} = n \sum_0^n u_\nu - \sum_0^n \nu u_\nu = n U_n - \sum_0^n \nu u_\nu$, or

$$(2.2) \quad U_n = \frac{1}{n} U_n^{(1)} + \frac{1}{n} \sum_1^n \nu u_\nu, \quad n = 1, 2, \dots$$

We get similarly

$$(2.3) \quad U_n = \frac{1}{2n} U_{2n}^{(1)} + \left(\frac{1}{2n} U_{2n}^{(1)} - \frac{1}{n} U_n^{(1)} \right) - \frac{1}{n} \sum_{n+1}^{2n} (2n - \nu)u_\nu,$$

for $n = 1, 2, \dots$, and ([4], p. 186)

$$(2.4) \quad U_n = \frac{U_{n+\kappa}^{(1)}}{n + \kappa} + \frac{n}{\kappa} \left(\frac{U_{n+\kappa}^{(1)}}{n + \kappa} - \frac{U_n^{(1)}}{n} \right) - \frac{1}{\kappa} \sum_{\nu=n+1}^{n+\kappa} (n - \nu + \kappa)u_\nu, \quad \kappa \geq 1, n \geq 1.$$

This reduces to (2.3) for $\kappa = n$, whereas for $\kappa = 1$ it yields

$$(2.5) \quad U_n = \frac{1}{n + 1} U_{n+1}^{(1)} + \frac{1}{n + 1} \sum_1^n \nu u_\nu.$$

We are using in addition the fact that the assumption

$$(2.6) \quad -1 \leq f(x) \leq 1 \quad \text{for} \quad -\pi < x < \pi$$

involves

$$(2.7) \quad \left| \sum_0^{n-1} s_\nu(x) \right| \leq n, \quad n = 1, 2, \dots,$$

which is the classical result of Fejér, and (compare [9]) the relations

$$(2.8) \quad \begin{aligned} |\sigma_{2n}(x) - \sigma_n(x)| &\leq 2/\pi, \\ |\bar{\sigma}_{2n}(x) - \bar{\sigma}_n(x)| &\leq 2/\pi, \quad n = 1, 2, \dots, \end{aligned}$$

where $\sigma_n = (1/n)\sum_0^{n-1} s_\nu$, and $\bar{\sigma}_n$ is the corresponding mean for the conjugate series. In view of (2.1) the last inequalities can be written as

$$(2.9) \quad \begin{aligned} &\left| \sum_1^n \nu(a_\nu \cos \nu x + b_\nu \sin \nu x) \right. \\ &\quad \left. + \sum_{n+1}^{2n} (2n - \nu)(a_\nu \cos \nu x + b_\nu \sin \nu x) \right| \leq \frac{4}{\pi} n, \\ &\left| \sum_1^n \nu(b_\nu \cos \nu x - a_\nu \sin \nu x) \right. \\ &\quad \left. + \sum_{n+1}^{2n} (2n - \nu)(b_\nu \cos \nu x - a_\nu \sin \nu x) \right| \leq \frac{4}{\pi} n. \end{aligned}$$

We shall use also the following more general inequalities:

$$\begin{aligned} &\left| \sum_{\nu=1}^n \nu \{ a_{\nu+\lambda-1} \cos (\nu+\lambda-1)x + b_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \} \right. \\ &\quad \left. + \sum_{\nu=n+1}^{2n} (2n-\nu) \{ a_{\nu+\lambda-1} \cos (\nu+\lambda-1)x + b_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \} \right| \leq \frac{4}{\pi} n, \\ &\left| \sum_{\nu=1}^n \nu \{ b_{\nu+\lambda-1} \cos (\nu+\lambda-1)x - a_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \} \right. \\ &\quad \left. + \sum_{\nu=n+1}^{2n} (2n-\nu) \{ b_{\nu+\lambda-1} \cos (\nu+\lambda-1)x - a_{\nu+\lambda-1} \sin (\nu+\lambda-1)x \} \right| \leq \frac{4}{\pi} n, \end{aligned}$$

$\lambda \geq 1 - n/2.$

3. Two auxiliary theorems. We prove the following lemma:

LEMMA 1. *The sequence $\rho_n = n^{-1} \sum_{n+1}^{2n} (2n - \nu)/\nu$, ($n = 1, 2, 3, \dots$), is monotonically increasing, and $\rho_n \uparrow 2 \log 2 - 1 = 0.38629 \dots$.*

For the proof note that

$$\begin{aligned} \rho_{n+1} - \rho_n &= 2 \left(\sum_{n+2}^{2n+2} \frac{1}{\nu} - \sum_{n+1}^{2n} \frac{1}{\nu} \right) = 2 \left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \right) \\ &= \frac{2}{2n+1} - \frac{1}{n+1} > 0; \end{aligned}$$

furthermore

$$\rho_n = \frac{1}{n} \sum_{n+1}^{2n} \frac{2 - \nu/n}{\nu/n} \rightarrow \int_1^2 \frac{2 - x}{x} dx = 2 \log 2 - 1.$$

LEMMA 2. *The sequence $\tau_n = n^{-1} \sum_n^{2n-1} (\nu - n)/\nu$, ($n = 1, 2, 3, \dots$), is monotonically increasing, and*

$$\tau_n \uparrow 1 - \log 2 = 0.30685 \dots$$

Indeed

$$\tau_{n+1} - \tau_n = \sum_n^{2n-1} \frac{1}{\nu} - \sum_{n+1}^{2n+1} \frac{1}{\nu} = \frac{1}{n} - \frac{1}{2n} - \frac{1}{2n+1} = \frac{1}{2n} - \frac{1}{2n+1} > 0$$

and

$$\tau_n = \frac{1}{n} \sum_n^{2n-1} \frac{\nu/n - 1}{\nu/n} \rightarrow \int_1^2 \frac{x - 1}{x} dx = 1 - \log 2.$$

4. **Estimates for $\sum \nu |a_\nu|$, $\sum \nu |b_\nu|$, and related sums.** In view of (2.1) the inequalities (2.8) can be written as

$$\left| \sum_1^n \nu u_\nu + \sum_{n+1}^{2n} (2n - \nu) u_\nu \right| \leq \frac{4}{\pi} n, \quad n = 1, 2, \dots,$$

where u_ν is $a_\nu \cos \nu x + b_\nu \sin \nu x$ or $b_\nu \cos \nu x - a_\nu \sin \nu x$, respectively. In particular, for $x = 0$, we have $\sum_1^n \nu a_\nu + \sum_{n+1}^{2n+1} (2n - \nu) a_\nu \leq 4n/\pi$, ($n \geq 2$), and $|a_1| \leq 4/\pi$; hence

$$(4.1) \quad \sum_1^n (\nu a_\nu + p) + \sum_{n+1}^{2n-1} (2n - \nu) \left(a_\nu + \frac{p}{\nu} \right) \leq n \left(\frac{4}{\pi} + p \right) + p \sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu}, \quad n \geq 2.$$

We now assume $p > 0$ and

$$(4.2) \quad \nu a_\nu \geq -p, \quad \nu = 1, 2, \dots, 2n - 1;$$

then (4.1) gives

$$\sum_1^n (\nu a_\nu + p) \leq n \left\{ \frac{4}{\pi} + p + \frac{p}{n} \sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu} \right\},$$

and, using Lemma 1, we obtain $\sum_1^n \nu a_\nu < n \{ 4/\pi + p(2 \log 2 - 1) \}$, ($n \geq 2$). Also from (4.2) $\nu(|a_\nu| - a_\nu) \leq 2p$, ($\nu = 1, 2, \dots, 2n - 1$); hence $\sum_1^n \nu |a_\nu| < n \{ 4/\pi + p(1 + 2 \log 2) \}$, ($n \geq 1$). In the same way the assumption $\nu b_\nu \geq -p$, ($\nu = 1, 2, \dots, 2n - 1$), yields $\sum_1^n \nu |b_\nu| < n \{ 4/\pi + p(1 + 2 \log 2) \}$. A fortiori

$$(4.3) \quad \left| \sum_1^n \nu a_\nu \cos \nu x \right| < n \left\{ \frac{4}{\pi} + p(1 + 2 \log 2) \right\}, \quad n \geq 1,$$

and

$$(4.4) \quad \left| \sum_1^n \nu b_\nu \sin \nu x \right| < n \left\{ \frac{4}{\pi} + p(1 + 2 \log 2) \right\}, \quad n \geq 1.$$

Again from (4.1)

$$\sum_{n+1}^{2n-1} (2n - \nu) \left(a_\nu + \frac{p}{\nu} \right) \leq n \left(\frac{4}{\pi} + p \right) + p \sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu}, \quad n \geq 2;$$

hence $\sum_{n+1}^{2n-1} (2n - \nu) a_\nu \leq n(4/\pi + p)$. But from (4.2) we have $(2n - \nu) \cdot (|a_\nu| - a_\nu) \leq 2p(2n - \nu)/\nu$; hence

$$(4.5) \quad \sum_{n+1}^{2n-1} (2n - \nu) |a_\nu| \leq n \left(\frac{4}{\pi} + p \right) + 2p \sum_{n+1}^{2n-1} \frac{2n - \nu}{\nu}.$$

Using again Lemma 1, we get

$$(4.6) \quad \sum_{n+1}^{2n-1} (2n - \nu) |a_\nu| < n \left\{ \frac{4}{\pi} + p(4 \log 2 - 1) \right\},$$

and similarly $\sum_{n+1}^{2n-1} (2n - \nu) |b_\nu| < n \{4/\pi + p(4 \log 2 - 1)\}$. Summarizing, we have the following theorem:

THEOREM 1. *Suppose $|f(x)| \leq 1$; if in addition*

$$(4.7) \quad \nu a_\nu \geq -p, \quad \nu = 1, 2, \dots, 2n - 1; p > 0; n > 0,$$

then, for $\kappa = 1, \dots, n$,

$$(4.8) \quad \sum_1^\kappa \nu |a_\nu| < \kappa \left\{ \frac{4}{\pi} + p(1 + 2 \log 2) \right\},$$

$$(4.9) \quad \sum_{\nu=\kappa+1}^{2\kappa} (2\kappa - \nu) |a_\nu| < \kappa \left\{ \frac{4}{\pi} + p(4 \log 2 - 1) \right\}.$$

Similarly, if

$$(4.10) \quad \nu b_\nu \geq -p, \quad \nu = 1, \dots, 2n - 1; p > 0; n > 0,$$

then

$$(4.11) \quad \sum_1^\kappa \nu |b_\nu| < \kappa \left\{ \frac{4}{\pi} + p(1 + 2 \log 2) \right\},$$

$$(4.12) \quad \sum_{\nu=\kappa+1}^{2\kappa} (2\kappa - \nu) |b_\nu| < \kappa \left\{ \frac{4}{\pi} + p(4 \log 2 - 1) \right\}.$$

For $\kappa = n = 1$ we have also the inequalities $|a_1| \leq 4/\pi, |b_1| \leq 4/\pi$. Note that $4/\pi = 1.2732 \dots, 1 + 2 \log 2 = 2.3862 \dots, 4 \log 2 - 1 = 1.7725 \dots$.

5. Estimates for partial sums. We now assume (4.7) and (4.10); applying Theorem 1, formula (2.2), and the inequality (2.7) we obtain

$$(5.1) \quad \left| \frac{a_0}{2} + \sum_1^\kappa a_\nu \cos \nu x \right| < 1 + \frac{4}{\pi} + p(1 + 2 \log 2),$$

for $\kappa = 1, \dots, n$, and

$$(5.2) \quad \left| \sum_1^\kappa b_\nu \sin \nu x \right| < 1 + \frac{4}{\pi} + p(1 + 2 \log 2), \quad \kappa = 1, \dots, n.$$

Using again (2.2) with $u_\nu = a_\nu \cos \nu x + b_\nu \sin \nu x$ we obtain $|s_\kappa(f; x)| < 1 + 8/\pi + 2p(1 + 2 \log 2), (\kappa = 1, \dots, n)$. Using now (1.8) and the assumption (1.4) we get

$$|s_\kappa(f; x)| < (1 + 8/\pi)\mu + 2p(1 + 2 \log 2) < 3.547\mu + 4.773p.$$

This is sharper than (1.6).

Note that the formula $s_0 = a_0/2 = (2\pi)^{-1} \int_{-\pi}^\pi f(x) dx$ gives $|s_0| \leq \mu$, assuming (1.4) only. In a similar way formulas (2.3), (2.8), (4.9), and (4.12) give

$$(5.3) \quad \left| \frac{a_0}{2} + \sum_1^\kappa a_\nu \cos \nu x \right| < 1 + \frac{6}{\pi} + p(4 \log 2 - 1),$$

$$(5.4) \quad \left| \sum_1^\kappa b_\nu \sin \nu x \right| < 1 + \frac{6}{\pi} + p(4 \log 2 - 1),$$

$$|s_\kappa(f; x)| < 1 + \frac{10}{\pi} + 2p(4 \log 2 - 1),$$

$\kappa = 1, \dots, n$.

Since $1 + 10/\pi = 4.18309 \dots, 8 \log 2 - 2 = 3.54518 \dots$, this estimate again is sharper than (1.6).

Using again formula (1.8) and summarizing, we have the following theorem:

THEOREM 2. *If $|f(x)| \leq \mu$ and*

$$(5.5) \quad \nu a_\nu \geq -p, \quad \nu b_\nu \geq -p, \quad \nu = 1, \dots, 2n - 1; p > 0,$$

then for $\kappa = 1, \dots, n$

$$\left| \frac{a_0}{2} + \sum_1^\kappa (a_\nu \cos \nu x + b_\nu \sin \nu x) \right| < \begin{cases} (1 + 8/\pi)\mu + 2p(1 + 2 \log 2) \\ (1 + 10/\pi)\mu + 2p(4 \log 2 - 1). \end{cases}$$

6. **Estimates for $\sum_{\nu=l+1}^{2l-1} (2l-\nu) |a_{\nu+\lambda-1}|$ and related sums.** In [9] the author proved, under the assumption (2.6), that

$$\left| \sum_{\nu=1}^l \nu u_{\nu+\lambda-1} + \sum_{\nu=l+1}^{2l} (2l-\nu) u_{\nu+\lambda-1} \right| \leq \frac{4}{\pi} l, \quad \lambda = 1, 2, 3, \dots; l \geq 1;$$

here u_ν stands for $a_\nu \cos \nu x + b_\nu \sin \nu x$ or $b_\nu \cos \nu x - a_\nu \sin \nu x$. Hence, for $l > 1$, $\sum_{\nu=1}^l \nu a_{\nu+\lambda-1} + \sum_{\nu=l+1}^{2l-1} (2l-\nu) a_{\nu+\lambda-1} \leq 4l/\pi$, and for $l=1$, $|u_\lambda| \leq 4/\pi$, ($\lambda=1, 2, 3, \dots$). Thus for $l > 1$

$$\begin{aligned} (6.1) \quad & \sum_{\nu=1}^l \nu \left(a_{\nu+\lambda-1} + \frac{p}{\nu + \lambda - 1} \right) + \sum_{\nu=l+1}^{2l-1} (2l-\nu) \left(a_{\nu+\lambda-1} + \frac{p}{\nu + \lambda - 1} \right) \\ & \leq \frac{4}{\pi} l + p \left\{ \sum_{\nu=1}^l \frac{\nu}{\nu + \lambda - 1} + \sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu + \lambda - 1} \right\}. \end{aligned}$$

We now assume for a given integer N that

$$(6.2) \quad \nu a_\nu \geq -p, \quad \nu = 1, 2, \dots, N,$$

and suppose that

$$(6.3) \quad 2l + \lambda - 2 \leq N;$$

then from (6.1), for $l > 1$, we have

$$(6.4) \quad \sum_{\nu=l+1}^{2l-1} (2l-\nu) a_{\nu+\lambda-1} \leq \frac{4}{\pi} l + p \sum_{\nu=1}^l \frac{\nu}{\nu + \lambda - 1}.$$

Furthermore $|a_\nu| - a_\nu \leq 2p/\nu$, ($\nu=1, \dots, N$); hence using (6.4) we obtain

$$\sum_{\nu=l+1}^{2l-1} (2l-\nu) |a_{\nu+\lambda-1}| \leq \frac{4}{\pi} l + p \left\{ 2 \sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu + \lambda - 1} + \sum_{\nu=1}^l \frac{\nu}{\nu + \lambda - 1} \right\}.$$

Similarly, assuming

$$(6.5) \quad \nu b_\nu \geq -p, \nu = 1, \dots, N,$$

we get, for $l > 1$,

$$\sum_{\nu=l+1}^{2l-1} (2l-\nu) |b_{\nu+\lambda-1}| \leq \frac{4}{\pi} l + p \left\{ 2 \sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu + \lambda - 1} + \sum_{\nu=1}^l \frac{\nu}{\nu + \lambda - 1} \right\}.$$

A simple calculation yields

$$\sum_{\nu=l+1}^{2l-1} \frac{2l-\nu}{\nu + \lambda - 1} \leq \frac{1}{l + \lambda} \sum_{\nu=l+1}^{2l-1} (2l-\nu) = \frac{l(l-1)}{2(l+\lambda)},$$

$$\sum_{\nu=1}^l \frac{\nu}{\nu + \lambda - 1} < \frac{l^2}{l + \lambda - 1}.$$

We have thus proved the following theorem:

THEOREM 3. *Under the assumptions (2.6), (6.2), (6.3), and (6.5) we have, for $l > 1$, $\lambda = 1, 2, 3, \dots$,*

$$(6.6) \quad \sum_{\nu=l+1}^{2l} (2l - \nu)v_{\nu+\lambda-1} < \left\{ \frac{4}{\pi} + p \left(\frac{l-1}{l+\lambda} + \frac{l}{l+\lambda-1} \right) \right\} l,$$

where v_ν stands for $|a_\nu|$ or $|b_\nu|$.

Formula (6.6) remains true also for $l = 1$.

7. Further estimates for partial sums. Assuming (2.6) we get, from (2.4) and (2.7),

$$(7.1) \quad |U_n| \leq 1 + \frac{2n}{\kappa} + \frac{1}{\kappa} \left| \sum_{\nu=n+1}^{n+\kappa} (n - \nu + \kappa)u_\nu \right|, \quad \kappa \geq 1, \quad n \geq 1,$$

where u_ν means $a_\nu \cos \nu x$, $b_\nu \sin \nu x$, or $a_\nu \cos \nu x + b_\nu \sin \nu x$. Replacing in (6.6) l by κ and λ by $n - \kappa + 1$, we get

$$\sum_{\nu=l+1}^{2l} (2l - \nu)v_{\nu+\lambda-1} = \sum_{\nu=\kappa+1}^{2\kappa} (2\kappa - \nu)v_{\nu+n-\kappa} \equiv \sum_{\nu=n+1}^{n+\kappa} (n - \nu + \kappa)v_\nu;$$

hence, using (6.6) and (7.1), we obtain

$$(7.2) \quad |U_n| < 1 + \frac{2n}{\kappa} + \frac{4}{\pi} + p \left(\frac{\kappa - 1}{n + 1} + \frac{\kappa}{n} \right), \quad n \geq \kappa \geq 1.$$

The condition (6.3) becomes

$$(7.3) \quad n + \kappa - 1 \leq N.$$

Here U_n means $a_0/2 + \sum_1^n a_\nu \cos \nu x$ or $\sum_1^n b_\nu \sin \nu x$. We observe that $2n/\kappa + p\{(\kappa - 1)/(n + 1) + \kappa/n\}$ decreases with increasing κ where $\kappa(\kappa + 1) \leq 2n^2(n + 1)/p(2n + 1)$. We thus put

$$(7.4) \quad \kappa = \left[\frac{1}{2} + \left\{ \frac{1}{4} + \frac{2n^2(n + 1)}{p(2n + 1)} \right\}^{1/2} \right];$$

obviously

$$\kappa > -\frac{1}{2} + \left\{ \frac{1}{4} + \frac{2n^2(n + 1)}{p(2n + 1)} \right\}^{1/2} \geq 0;$$

hence $\kappa \geq 1$. We now get easily the relation

$$\frac{2n}{\kappa} + p \frac{\kappa}{n} < \frac{2n}{-\frac{1}{2} + \left\{ \frac{1}{4} + \frac{2n^2(n+1)}{p(2n+1)} \right\}^{1/2}} + \frac{p}{n} \left(-\frac{1}{2} + \left\{ \frac{1}{4} + \frac{2n^2(n+1)}{p(2n+1)} \right\}^{1/2} \right),$$

and

$$\frac{2n}{\kappa} + p \frac{\kappa}{n} + p \frac{\kappa - 1}{n + 1} < \frac{2p(2n + 1)}{n(n + 1)} \left\{ \frac{1}{4} + \frac{2n^2(n + 1)}{p(2n + 1)} \right\}^{1/2} \leq 4p^{1/2}$$

if

$$(7.5) \quad p \leq 16n/9, \quad p \leq 2n^4/(2n + 1)^2.$$

On putting $N = 2n - 1$ in (7.3) we get $\kappa \leq n$; in view of (7.4) this condition is satisfied if

$$(7.6) \quad p \geq \frac{2n(n + 1)}{(2n + 1)(n - 1)}, \quad n > 1.$$

Thus with the restrictions (7.5) and (7.6) the assumptions (2.6) and (5.5) imply

$$(7.7) \quad |U_n| < 1 + 4/\pi + 4p^{1/2}.$$

In the cases $p > 16n/9$ or $p > 2n^4/(2n + 1)^2$, (7.7) can be deduced from $a_0^2/2 + \sum_1^n (a_\nu^2 + b_\nu^2) \leq \pi^{-1} \int_{-\pi}^\pi f^2(x) dx \leq 2$. Now

$$\begin{aligned} \frac{1}{2} |a_0| + \sum_1^n |a_\nu| &\leq \left\{ \frac{1}{2} a_0^2 + \sum_1^n a_\nu^2 \right\}^{1/2} \left(\frac{1}{2} + n \right)^{1/2} \\ &\leq (1 + 2n)^{1/2} < 1 + 4/\pi + 4p^{1/2}. \end{aligned}$$

Finally in the case $p < 2n(n + 1)/(2n + 1)(n - 1)$ we can deduce (7.7) from (2.5), (2.7), (4.8), and (4.11), which give $|U_n| < 1 + [n/(n + 1)] \cdot \{4/\pi + p(1 + 2 \log 2)\}$ and this is in the present case less than $1 + 4/\pi + 4p^{1/2}$ for $n > 1$, since $[n/(n + 1)]p^{1/2} < 4/(1 + 2 \log 2)$. Summarizing and using (1.8), we complete the proof of the following theorem:

THEOREM 4. *Under the assumptions (1.4) and (5.5),*

$$\begin{aligned} \left| \frac{a_0}{2} + \sum_1^\kappa a_\nu \cos \nu x \right| &< \left(1 + \frac{4}{\pi} \right) \mu + 4(p\mu)^{1/2}, \\ \left| \sum_1^\kappa b_\nu \sin \nu x \right| &< \left(1 + \frac{4}{\pi} \right) \mu + 4(p\mu)^{1/2}, \quad \kappa = 1, 2, \dots, n. \end{aligned}$$

In the case $u_\nu = a_\nu \cos \nu x + b_\nu \sin \nu x$, we get from (7.1) and (6.6) in a similar way

$$|U_n| < 1 + \frac{2n}{\kappa} + \frac{8}{\pi} + 2p \left(\frac{\kappa - 1}{n + 1} + \frac{\kappa}{n} \right), \quad 1 \leq \kappa \leq n.$$

We now put

$$\kappa = \left[\frac{1}{2} + \left\{ \frac{1}{4} + \frac{n^2(n+1)}{p(2n+1)} \right\}^{1/2} \right];$$

that is, in our previous argument p is replaced by $2p$. We get the following theorem:

THEOREM 5. *Under the assumptions (1.4) and (5.5),*

$$|s_\kappa(f; x)| < \left(1 + \frac{8}{\pi} \right) \mu + 4(2p\mu)^{1/2}, \quad \kappa = 1, \dots, n.$$

Since $1 + 8/\pi = 3.5464 \dots < 5$, and $4(2)^{1/2} < 8$, our estimate is sharper than (1.7).

The results can be extended to almost periodic functions (for $p=0$ see [3], [7], [8], [9]) and to Fourier integrals; also the condition (1.3) can be generalized to "slow oscillation," improving upon some results of Fekete and the author.

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