EXPANSION OF FUNCTIONS IN SOLUTIONS OF FUNCTIONAL EQUATIONS*

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1. Introduction. In analysis a number of functional equations have solutions of the form

\[ x^r \sum_{s=0}^{\infty} \alpha_s r^s. \]

Examples are (a) linear differential equations with a regular singular point at the origin, (b) the Volterra homogeneous integral equation with a regular singularity, (c) the linear q-difference equation, (d) the Fuchsian equation of infinite order. There are many others including mixed q-difference and differential equations.

Consider the equation

\[ L(x, \lambda) \rightarrow y = 0 \]

where \( \lambda \) is a parameter and \( L(x, \lambda) \) is an operator with the following property:

\[ L(x, \lambda) \rightarrow x^p = x^p f(x, p, \lambda) = x^p \sum_{\mu=0}^{\infty} f_\mu(p, \lambda) x^\mu, \]

the series converging for \( |x| \leq N < r \) for all values of \( p \), which may be a complex number. The purpose of this paper is to consider under what conditions a set of values \( \{ \lambda_m \} \), \( (m = 0, 1, 2, \cdots) \), can be determined so that for \( \lambda = \lambda_m \) there will exist a solution of the form

\[ y_{m+\sigma}(x) = x^{m+\sigma} \sum_{s=0}^{\infty} \alpha_s (m+\sigma)^s x^s = x^{m+\sigma} \sum_{s=0}^{\infty} \alpha_s (m+\sigma)^s x^s \]

\[ = x^{m+\sigma} \left\{ a_0 (m+\sigma)^s h_m(x) \right\} \]

such that an arbitrary function \( x^\sigma f(x), f(x) \) being analytic for \( |x| < \rho \), can be expanded in a series

\[ x^\sigma f(x) = \sum_{m=0}^{\infty} a_m y_{m+\sigma}(x) \]

which converges and represents the function in some region. For the

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sake of simplicity we will consider \( \sigma = 0 \); the extension to \( \sigma \) any number will be evident. In order to do this we will make use of the following theorem due to Mrs. Gertrude Stith Ketchum* which will be denoted by Theorem K:

**Theorem K.** Consider the set of functions \( g_m(x) = x^m \{ 1 + h_m(x) \} \) where \( h_m(0) = 0 \) and \( h_m(x) \) is analytic for \( |x| < R \). If there exists an \( M_{N,m} \) such that \( |h_m(x)| \leq M_{N,m} \) for a given positive \( N < R \) and \( |x| \leq N \), and if \( \lim \sup M_{N,m} = K_N \) (finite), then any function analytic for \( |x| < p \) has a unique uniformly convergent expansion \( f(x) = \sum_{m=0}^{\infty} \alpha_m g_m(x) \) for \( |x| \leq R < G \), where

\[
G = \min \left \{ p, \max_N \left \{ N + K_N \right \}^{-1} \right \},
\]

and the expansion converges absolutely for \( |x| < G \).

We seek then to determine conditions under which the \( y_m(x) \), \( (m = 0, 1, \cdots) \), will satisfy the conditions of this theorem.

2. **Sufficient conditions.** Operating formally upon both sides of (4) \((\sigma = 0)\) with the operator \( L(x, \lambda) \) we get

\[
L(x, \lambda) y_m(x) = \sum_{s=0}^{\infty} \alpha_s^{(m)} x^{m+s} \sum_{\mu=0}^{\infty} f_\mu(m + s, \lambda) x^\mu = 0.
\]

Equating the coefficients of powers of \( x \) to zero we get the following set of equations for the determination of the \( \alpha_s^{(m)} \), \((s = 0, 1, \cdots)\):

\[
\begin{align*}
\alpha_0^{(m)} f_0(m, \lambda) &= 0, \\
\alpha_1^{(m)} f_0(m + 1, \lambda) + \alpha_0^{(m)} f_1(m, \lambda) &= 0, \\
&\vdots \\
\alpha_s^{(m)} f_0(m + s, \lambda) + \alpha_{s-1}^{(m)} f_1(m + s, \lambda) + \cdots + \alpha_0^{(m)} f_{s+1}(m, \lambda) &= 0, \\
&\vdots
\end{align*}
\]

If \( \alpha_0^{(m)} \neq 0 \), then \( f_0(m, \lambda) = 0 \) to give a solution of the desired form. Suppose that

\[
f_0(m, \lambda_m) = 0, \quad m = 0, 1, \cdots,
\]

determines a set of characteristic values \( \{ \lambda_m \} \), and further suppose that

\[
f_0(p, \lambda_m) \neq 0, \quad m \neq p.
\]

The coefficients $\alpha_{s+1}^{(m)}$ can be determined for $s = 0, 1, \cdots$. Since $\alpha_0^{(m)}$ is arbitrary, we will choose it to be unity. By the method of Frobenius* we get the following set of inequalities:

$$
A_{s+1}^{(m)} \leq A_s^{(m)} \left\{ \begin{array}{c}
M_N(m + s, \lambda_m) + |f_0(m + s, \lambda_m)| \\
|f_0(m + s + 1, \lambda_m)|
\end{array} \right\}
= A_s^{(m)} P(m, s),
$$

where

$$
A_{s+1}^{(m)} = \{ | \alpha_s^{(m)} | M_N(m + s, \lambda_m) + | \alpha_{s-1}^{(m)} | M_N(m + s - 1) N^{-1} + \cdots
+ | \alpha_0^{(m)} | M_N(s) N^{-s} \} | f_0(m + s + 1, \lambda_m) |^{-1}
$$

and $M_N(m + s, \lambda_m)$ are such that

$$
\frac{d}{dx} f(x, m + s, \lambda_m) \leq M_N(m + s, \lambda_m).
$$

It is evident that

$$
| h_m(x) | \leq F_m(x),
$$

where

$$
F_m(x) = \sum_{s=1}^{\infty} A_s^{(m)} | x |^s.
$$

Suppose

(a) $\lim_{s \to \infty} \sup P(m, s) = P(m)$,

(b) $\lim_{m \to \infty} \sup P(m) = \rho$,

(c) $\lim_{m \to \infty} \sup P(m, s) = Q(s)$,

(d) $\lim_{s \to \infty} \sup Q(s) = q$.

Let $R$ be the smallest of $(P(m))^{-1}$, $(m = 0, 1, \cdots)$, $\rho^{-1}$, $q^{-1}$, and $N$. Then

$$
\lim_{s \to \infty} \sup A_{s+1}^{(m)}/A_s^{(m)} \leq P(m),
$$

and (14) converges for $|x| < R$. Since $N$ is at our choice, let $N$ be less than $R$. We have also

(17) \[ A_{s+1}^{(m)} \leq \prod_{i=0}^{s} P(m, i); \]
hence
\[ \limsup_{m \to \infty} A_{s+1}^{(m)} \leq \prod_{i=0}^{s} Q(i) = A_{s+1} \]
and
\[ \limsup_{s \to \infty} A_{s+1}/A_s \leq q. \]
Then the series
\[ F(x) = \sum_{s=0}^{\infty} A_s \cdot x^s \]
converges for \(|x| \leq N < R\).
Let \(M_N^{(m)}\) be such that \(|h_m(x)| \leq F_m(x) \leq M_N^{(m)}\) and \(M_N\) such that \(F(x) \leq M_N\); then
\[ \limsup_{m \to \infty} M_N^{(m)} = K_N \leq M_N. \]
The conditions of Theorem K are satisfied and we may state the following theorem:

**Theorem.** If we have a functional equation with an operator having the property (3), if there exists a set of values fulfilling conditions (8) and (9), and if conditions (15) are satisfied, then there exists a unique expansion of the form
\[ f(x) = \sum_{m=0}^{\infty} a_m y_m(x), \]
where \(f(x)\) is analytic for \(|x| < \rho\), which will converge uniformly for \(|x| \leq R < G\), where
\[ G = \min \left\{ \max_N N(1 + K_N)^{-1} \right\}. \]
The expansion converges and represents the function for \(|x| < G\).

3. **Examples.** Suppose we have the equation
\[ \sum_{j=0}^{n} P_{i,j}(x, \lambda) \delta^{n-j} y(x) + \sum_{i=1}^{r} \lambda^i \sum_{j=0}^{m} P_{i,j}(x, \lambda) \delta^{m-j} y(x) \]
\[ + \int_{0}^{x} g(x, t, \lambda) y(t) dt = 0, \]
where

\[ P_{0,0}(x) = 1, \quad \left| \frac{d}{dx} P_{j,i}(x, \lambda) \right| \leq M_{N,N}^{(j,i)}, \quad |x| \leq N < r, \quad |\lambda| > N, \]

\[ g(x, t, \lambda) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}(\lambda) x^i t^j, \]

\[ G(x, p, \lambda) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij}(\lambda) \frac{i + j + 1}{i + p + 1} x^{i+j}, \quad p \text{ an integer}, \]

and

\[ |G(x, p, \lambda)| \leq M_{N,N}, \quad |x| \leq N < r, \quad |\lambda| > N. \]

The function \( \delta y(x) \) is either \( x^d \frac{d^s y}{dx^s}, y(q,x) \) with \( |q_n| > 1 \) and \( |q_n - i| > |q_{n-1}|, (i=1, 2, \cdots, n) \), or \( y(q,x) \) with \( |q| > 1 \). The function \( f_0(m, \lambda) \) will be a polynomial of degree \( r \) in \( \lambda \) and of degree \( n \) in either \( q^m \) or \( m \), or a polynomial of degree \( m \) in \( q_j, (j=0, 1, \cdots, n) \). The conditions of the theorem can then be shown to be satisfied, and the expansion of an arbitrary function follows. Consider the case for which \( r = 1, m = 0, \) and \( P_{n,i}(x, \lambda), g(x, t, \lambda) \) are independent of \( x \). If \( \{\lambda_m\}, (m=0, 1, \cdots) \), is the set of characteristic values and \( y_m(x) \) are the corresponding functions, then the solution of the nonhomogeneous equation

\[ \delta^n y(x) + P_1(x) \delta^{n-1} y(x) + \cdots + P_0(x) y(x) \]

(21)

\[ + \int_0^x g(x, t) y(t) dt + \lambda y(x) = f(x), \]

where \( f(x) \) is analytic for \( |x| < \rho \), has a solution of the form

(22)

\[ y(x) = \sum_{m=0}^{\infty} \frac{a_m}{\lambda - \lambda_m} y_m(x), \quad \lambda \neq \lambda_m, \]

where \( f(x) = \sum_{m=0}^{\infty} a_m y_m(x). \) This is easily verified by substitution. If \( \lambda = \lambda_p \) and \( f(x) = x^{p+1} F(x) \), then the solution is of the form

\[ y(x) = \sum_{m=p+1}^{\infty} \frac{a_m}{\lambda_p - \lambda_m} y_m(x). \]

Consider the equation

(23)

\[ \sum_{n=0}^{\infty} \frac{A_n(x, \lambda)}{n!} \left( x \frac{d y}{dx} \right)^n + \lambda y(x) = 0, \]

where
\[ \frac{d}{dx} A_n(x, \lambda) \equiv 0, \quad n > n', \quad \left| \frac{d}{dx} A_n(x, \lambda) \right| \leq M_{N,N}^{(n)}, \]

\[ |x| < N < r, \quad |\lambda| > N, \]

\[ A_n(0, \lambda) \neq 0, \quad (n > n'), \quad A_n(0, \lambda) = a_n \text{ independent of } \lambda, \text{ and} \]

\[ \left( x \frac{dy}{dx} \right) \left( x \frac{dy}{dx} \right) = x \frac{dy}{dx}, \quad \left( x \frac{dy}{dx} \right)^2 = x \frac{d}{dx} \left\{ x \frac{dy}{dx} \right\}, \]

\[ \left( x \frac{dy}{dx} \right)^v = x \frac{d}{dx} \left\{ \left( x \frac{dy}{dx} \right)^{v-1} \right\}. \]

Then we obtain the relations

\[ f_0(m, \lambda) = \sum_{n=0}^{\infty} \frac{a_n m^n}{n!} + \lambda = 0, \quad \lambda = - \sum_{n=0}^{\infty} \frac{a_n m^n}{n!} = - f(m). \]

If \( a_n = 1, \lambda_m = -e^m. \)

This equation and others in which \( \lambda_m \) has the properties

\[ \lim_{m \to \infty} \sup_{s} \frac{|P(m+s)|}{|\lambda_{m+s+1} - \lambda_m|} = Q(s), \]

\[ \lim_{s \to \infty} Q(s) = q, \]

\[ \lim_{m \to \infty} \sup_{s} \frac{|P(m+s)|}{|\lambda_{m+s+1} - \lambda_m|} = \overline{P}(m), \]

\[ \lim_{m \to \infty} \overline{P}(m) = \bar{p}, \]

\( P(m+s) \) being a polynomial in \( m+s, \) will satisfy the conditions of the theorem, and the expansion follows.

The generalized Fuchsian equation

\[ \sum_{n=0}^{\infty} x^n A_n(x, \lambda) \frac{d^n y}{dx^n} + \lambda y(x) = 0 \]

is similar to the above except for the fact that the \( \lambda_m \) are given by Newton series.

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