A TEST-RATIO TEST FOR CONTINUED FRACTIONS*

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Introduction. The general question of convergence of continued fractions of the form \(1 + K_1 [b_n/1]\) remains in a large measure unanswered, even though continued fractions of this type are of especial importance from a function-theoretic point of view. Valuable contributions have been made by E. B. Van Vleck, A. Pringsheim, O. Szász, O. Perron, and others. Leighton and Wall [7] recently gave new types of convergence criteria for continued fractions of this kind. Jordan and Leighton in a paper to be published soon give a large number of new sets of sufficient conditions for convergence.

The purpose of the present paper is to establish the first test-ratio test for continued fractions and a very general theorem on convergence, which is also believed to be the first of its kind. This test leads to a class of continued fractions, the precise region of convergence of which is the interior of a circle. This is a new phenomenon.

1. A test-ratio test. Let

\[
1 + \frac{b_0}{1} = 1 + \frac{b_1}{1 + \frac{b_2}{1 + \ldots}}
\]

be a continued fraction in which the \(b_n\) are complex numbers \(\neq 0\).

**Theorem 1.** If the ratio \(|b_{n+1}/b_n|\) is less than or equal to \(k < 1\) for \(n\) sufficiently large, the continued fraction (1.1) converges at least in the wider sense. If \(|b_{n+1}/b_n|\) is greater than or equal to \(1/k > 1\) for \(n\) sufficiently large, the continued fraction diverges by oscillation. If the limit of the ratio is unity, the continued fraction may converge or diverge.

Suppose \(|b_{n+1}/b_n| \leq k < 1\) for \(n\) sufficiently large. It follows that there exists a positive integer \(N\) such that \(|b_n| < 1/4\) for \(n \geq N\). Each continued fraction \(K_1 [b_n/1]\) then converges (Van Vleck [2], Pringsheim [4]) for \(n \geq N\). The proof of the first statement of the theorem is complete.

Assume \(|b_{n+1}/b_n| \geq 1/k > 1\) for \(n\) sufficiently large. Write (1.1) in the equivalent form (Perron [8], p. 197)

\[
1 + \frac{b_0}{1} = 1 + \frac{b_1}{1 + \frac{b_2}{1 + \ldots}}
\]

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where
\[ a_{2n} = \frac{b_1b_3 \cdots b_{2n-1}}{b_3b_4 \cdots b_{2n}}, \quad a_{2n-1} = \frac{b_2b_4 \cdots b_{2n-2}}{b_1b_3 \cdots b_{2n-1}}, \quad n = 1, 2, 3, \ldots ; \quad b_0 = 1. \]

It will be shown that the series \( \sum |a_n| \) converges, and it will follow from a theorem of Stern [1] that the continued fraction (1.2), and hence (1.1), diverges by oscillation. It is sufficient to observe that
\[
|a_{2n}/a_{2n-2}| = |b_{2n-1}/b_{2n}| \leq k < 1,
\]
\[
|a_{2n+1}/a_{2n-1}| = |b_{2n}/b_{2n+1}| \leq k < 1,
\]
for \( n \) sufficiently large. Thus the two series \( \sum |a_{2n+1}| \) and \( \sum |a_{2n}| \) converge. It follows that the series \( \sum |a_n| \) converges, and the second statement of the theorem follows as indicated.

To prove the final statement of the theorem, it is sufficient to consider the example
\[
(1.3) \quad 1 + \frac{a}{1 + \frac{a}{1 + \cdots}}.
\]

When \( a = 1 \) it is well known that this continued fraction converges to the value \((1+5^{1/2})/2\). When \( a = -1 \), a computation of the successive approximants proves immediately that the continued fraction diverges. Indeed, Szász [6] has shown that the continued fraction (1.3) diverges for every \( \epsilon > 0 \), if \( a = -\epsilon - 1/4 \).

**Corollary.** If \( \lim_{n \to \infty} |b_{n+1}/b_n| = k \), the continued fraction (1.1) will converge, at least in the wider sense, if \( k < 1 \), and will diverge if \( k > 1 \).

The proof is immediate.

**Example.** A continued fraction with a circle as its region of convergence. Consider the continued fraction
\[
(1.4) \quad 1 + \frac{c_n x^n}{K_1[1 + c_{1} x]} = 1 + \frac{c_1 x}{1 + \frac{c_2 x^2}{1 + \cdots}},
\]
where the \( c_n \) are complex nonzero numbers. If \( \lim_{n \to \infty} |c_{n+1}/c_n| = c \neq 0 \), it follows from the preceding corollary that the continued fraction (1.4) converges, at least in the wider sense, to a function analytic except possibly for a finite number of poles in every closed region wholly interior to the circle \( |x| = 1/c \), and diverges outside. Further, if \( \lim_{n \to \infty} |c_{n+1}/c_n| = 0 \), (1.4) converges to a function meromorphic throughout the finite plane.
2. A general theorem on convergence. Leighton and Wall [7] gave an example of a convergent continued fraction (1.1) where the elements $b_n$ were everywhere dense in the complex plane. The following theorem attacks the general question of convergence from a different point of view. We assume as usual that all $b_n \neq 0$.

**Theorem 2.** Let $m_0, m_1, m_2, \ldots$ be any sequence of positive integers such that $m_0 = 2$, $m_{n+1} - m_n \geq 2$, ($n = 0, 1, 2, \ldots$). The numbers

$$
(2.1) \quad b_{m_0}, b_{m_1}, b_{m_2}, \ldots
$$

can be chosen in such a fashion that with at most one value in the complex plane excluded from each of the numbers $b_n$ not contained in the set (2.1), the continued fraction (1.1) will converge.

Let $A_n/B_n$ represent the $n$th approximant of (1.1), where $A_n$ and $B_n$ are given by the usual recursion relations

$$
(2.2) \quad A_n = A_{n-1} + b_n A_{n-2}, \quad B_n = B_{n-1} + b_n B_{n-2}, \quad n = 2, 3, 4, \ldots.
$$

By means of (2.2) write $A_j$ and $B_j$, ($j = 2, 3, \ldots, m_1 - 1$), as

$$
(2.2') \quad A_j = f_j A_0 + b_j A_1, \quad B_j = f_j B_0 + b_j B_1,
$$

where $f_j$ and $g_j$ are polynomials in the numbers $b_2, b_3, \ldots, b_{m_1 - 1}$ and do not depend on any other $b$’s. (Perron [8], p. 14, uses the symbol $A_{m_1 - m, m_r}$ for $f_j$, and $B_{m_1 - m, m_r}$ for $g_j$.) Suppose the numbers $f_j$ are nonzero. It is clear that $|b_{m_0}| = |b_2|$ can be chosen so small that simultaneously

$$
\left| \frac{A_j}{B_j} - \frac{A_1}{B_1} \right| < \frac{1}{2}, \quad j = 2, 3, \ldots, m_1 - 1.
$$

Now write $A_k$ and $B_k$, ($k = m_1, m_1 + 1, \ldots, m_2 - 1$), as

$$
(2.2'') \quad A_k = f_k A_{m_1 - 1} + b_k g_k A_{m_1 - 2}, \quad B_k = f_k B_{m_1 - 1} + b_k g_k B_{m_1 - 2},
$$

where $f_k$ and $g_k$ are polynomials in $b_{m_1 + 1}, b_{m_1 + 2}, \ldots, b_{m_2 - 1}$ and do not depend on any $b$’s not in this set. Similarly, let us suppose for the moment that the numbers $f_k$ are never zero. The number $|b_{m_1}|$ can then be taken so small that

$$
\left| \frac{A_k}{B_k} - \frac{A_{m_1 - 1}}{B_{m_1 - 1}} \right| < \frac{1}{2^k}, \quad k = m_1, m_1 + 1, \ldots, m_2 - 1.
$$
Continue the process. With the assumption that \( f_{t} \) is never zero it is clear that \(|b_{m_{t}}| \) can be chosen so small that

\[
\left| \frac{A_{t}}{B_{t}} - \frac{A_{m_{t}-1}}{B_{m_{t}-1}} \right| < \frac{1}{2^{r+1}}, \quad t = m_{r}, m_{r} + 1, \ldots, m_{r+1} - 1.
\]

The continued fraction will thus converge.

It remains to assign conditions to the numbers \( b_{n} \) so that the numbers \( f_{t} \) will be different from zero. It is sufficient to exclude precisely one value in the finite complex plane from each \( b_{n} \) not in the set \( b_{m_{0}}, b_{m_{1}}, \ldots \). For, in the general case, it follows from (2.2) that

\[
\begin{align*}
    f_{m_{r}} & = 1, \\
    f_{m_{r}+1} & = 1 + b_{m_{r}+2}, \\
    f_{m_{r}+s} & = f_{m_{r}+s-1} + b_{m_{r}+s}f_{m_{r}+s-2}, \quad s = 2, 3, \ldots, m_{r+1} - m_{r} - 1,
\end{align*}
\]

where \( f_{m_{r}+s-1} \) is a polynomial in \( b_{m_{r}+2}, b_{m_{r}+3}, \ldots, b_{m_{r}+s-1} \) and depends on no other \( b \)'s. The value \(-1\) is first excluded from \( b_{m_{r}+2} \). It follows from (2.3) that one value may be excluded from each successive \( b \) in such a way that \( f_{t} \) is never zero. This completes the proof of the theorem.

**BIBLIOGRAPHY**