CERTAIN INCOMPLETE NUMERICAL FUNCTIONS*

W. A. DWYER

1. Introduction. In a recent paper,† the author has shown that a certain pair of identities, given by Uspensky,‡ involving incomplete numerical functions in three variables result from a more general formula, which is obtained by the application of the method of paraphrase§ to a certain theta function identity. Uspensky's arithmetic proofs of his formulas are simple but give no suggestion of a systematic procedure for determining other identities of the same type. The analytic proof, however, provides a method by which related identities can be obtained by comparatively simple and straightforward operations.

The theta function identity (identity I, §2) mentioned above involves a product of the form

$$\theta_a(x + y + z)\phi_{\alpha\beta\gamma}(u, v),$$

where

$$\phi_{\alpha\beta\gamma}(u, v) = \theta_1' \frac{\theta_\alpha(u + v)}{\theta_\beta(u)\theta_\gamma(v)}$$

(α, β, γ are restricted to certain triads of the numbers 0, 1, 2, 3) is one of the sixteen doubly periodic functions of the second kind.‖ Sixty-four products of this type are possible, and each leads to a theta identity which, when paraphrased, yields a formula of the same general type as Uspensky's. Any one of these theta identities may be derived from any other one by elementary transformations (of the form $x = x + \pi/2 + \pi\tau/2$, for example) on one or more of the arguments. However, the transformations involving $\pi\tau/2$ are awkward. This suggests a basic set of identities from which all others may be obtained by comparatively easy transformations involving $\pi/2$ only.

* Much of the material appearing in this paper is summarized in an abstract entitled *On certain pseudo-periodic functions*, presented to the Society, April 10, 1937.

† American Journal of Mathematics, vol. 59 (1937), pp. 290–294.

‡ J. V. Uspensky, *Sur les relations entre les nombres des classes des formes quadratiques binaires et positives*, Bulletin de l'Académie des Science de l'URSS, 1925–1926. Also, American Journal of Mathematics, vol. 50 (1928), pp. 93–122; this Bulletin, vol. 36 (1930), pp. 743–754. The formulas mentioned above are labeled (III) and (IV) in this last paper.

§ E. T. Bell, Transactions of this Society, vol. 22 (1921), pp. 1–30; 198–219.

‖ Hermite's nomenclature, *Oeuvres*, vol. 4, pp. 199–200.

In this paper, we shall concern ourselves with this basic set of eight related theta identities and their paraphrases.

2. **The fundamental theta identities.** The procedure* used in establishing the theta identities appears in the proof of our identity I in the first paper; so only the results will be given here. We shall introduce the notation

$$\chi(x, y) = \sum_{r=-\infty}^{\infty} q^{r^2} e^{-2iry} \cot (x - r\pi\tau),$$

$$\chi_1(x, y) = \sum_{m=-\infty}^{\infty} q^{m^2/4} e^{-miy} \cot (x - m\pi\tau/2),$$

$$\chi_2(x, y) = \sum_{r=-\infty}^{\infty} q^{r^2} e^{-2iry} \csc (x - r\pi\tau),$$

$$\chi_3(x, y) = \sum_{m=-\infty}^{\infty} q^{m^2/4} e^{-miy} \csc (x - m\pi\tau/2),$$

where r is arbitrary and integral and m is odd.

Then the eight fundamental identities are as follows:

I. $\theta_3(x+y+z)\phi_{111}(x+y, -y) = \theta_3(z)\chi(x+y, x+z) - \theta_3(x+z)\chi(y, z)$.

II. $\theta_2(x+y+z)\phi_{111}(x+y, -y) = \theta_2(z)\chi_2(x+y, x+z) - \theta_2(x+z)\chi_2(y, z)$.

III. $\theta_3(x+y+z)\phi_{100}(x+y, -y) = \theta_2(z)\chi_1(x+y, x+z) - \theta_2(x+z)\chi_1(y, z)$.

IV. $\theta_2(x+y+z)\phi_{100}(x+y, -y) = \theta_3(z)\chi_3(x+y, x+z) - \theta_3(x+z)\chi_3(y, z)$.

V. $\theta_2(x+y+z)\phi_{001}(x+y, -y) = \theta_3(z)\chi_1(x+y, x+z) - \theta_2(x+z)\chi(y, z)$.

VI. $\theta_3(x+y+z)\phi_{001}(x+y, -y) = \theta_2(z)\chi_3(x+y, x+z) - \theta_3(x+z)\chi_2(y, z)$.

VII. $\theta_2(x+y+z)\phi_{010}(x+y, -y) = \theta_2(z)\chi(x+y, x+z) - \theta_3(x+z)\chi_1(y, z)$.

VIII. $\theta_3(x+y+z)\phi_{010}(x+y, -y) = \theta_3(z)\chi_2(x+y, x+z) - \theta_2(x+z)\chi_3(y, z)$.

The fifty-six remaining identities of this type may be obtained by means of elementary transformations, of the form $x \to x+\pi/2$, applied to one or more of the variables.

For completeness we shall list the sixty-four expressions and the transformations by which any one may be obtained from some one of the given identities. Let the transformations representing an increase of $\pi/2$ in the respective variables be designated as follows:

* The method is essentially that given by Appell for his theory of the doubly periodic functions of the third kind, as has been shown by M. A. Basoco, Acta Mathematica, vol. 57, pp. 95–100. Cf. also, M. A. Basoco, American Journal of Mathematics, vol. 54 (1932), pp. 242–252.

(a) z; (b) x; (c) y; (d) x and z; (e) y and z; (f) x and y; (g) x, y, and z.

By means of this notation the results may be given in a table:

$\phi_{\alpha\beta\gamma}$	θ_1		θ_2	θ_3		θ_0
111	a		0	0		a
212	g		f	f		g
221	b	II	d	d	I	b
122	c		e	e		c
100	a		0	0		a
203	g		f	f		g
230	b	IV	d	d	III	b
133	c		e	e		c
001	a		0	0		a
302	g		f	f		g
331	b	V	d	d	VI	b
032	c		e	e		c
010	a		0	0		a
313	g		f	f		g
320	b	VII	d	d	VIII	b
023	c		e	e		c

Thus, for example, $\theta_1(x+y+z)\theta_{133}(x+y, -y)$ can be obtained by applying transformation (c) (that is, letting y be increased by $\pi/2$) to identity IV.

3. **The paraphrases of the eight identities.** The formulas and the procedure made use of in the paraphrase of I are set forth in the earlier paper, and since the arithmetized expressions needed in the paraphrase of the remaining identities are available in the literature,* the final results only will be given.

* For arithmetized forms of the theta and phi functions, see, for example, E. T. Bell, Messenger of Mathematics, vol. 54 (1924). The arithmetized forms of cot $(x-n\pi\tau)$, and so on, may be obtained from the exponential forms but are given directly in a paper by M. A. Basoco, American Journal of Mathematics, vol. 54 (1932). The trigonometric reduction formulas may be found in E. T. Bell's *Arithmetical paraphrases*, Transactions of this Society, vol. 22 (1921), pp. 198–219, or built up from formulas appearing in this paper; see also Bell's paper in the Giornale di Matematiche, vol. 61 (1923).

With regard to the notation, the finite summations appearing in each identity refer to all integral solutions of the corresponding partitions listed with that identity, subject to the following conditions: i, h arbitrary; $\mu \gtrless 0$ odd; d, δ, t, τ, m, n, r, s, $k>0$, m, τ odd; and

$$\begin{aligned} &0 < \Delta_0 < \Delta_0', && \Delta_0' - \Delta_0 \equiv 0 \pmod 2, \\ &0 < \Delta_1 < \Delta_1', && \Delta_1' - \Delta_1 \equiv 1 \pmod 2, \\ &0 < \tau_0 < t_0, && t_0 - \tau_0 \equiv 0 \pmod 4, \\ &0 < \tau_2 < t_2, && t_2 - \tau_2 \equiv 2 \pmod 4. \end{aligned}$$

Also, $e(n)=1$ or 0 according as n is or is not a perfect square, and $a(n)=1$ or 0 according as n is or is not the sum of two squares.

Several of the identities given below, together with their partitions, could quite obviously be simplified, but it is believed best to present them in the form resulting directly from the paraphrase.*

In all cases $F(x, y, z)$ is a function defined for integral values of the arguments and subject to the parity conditions;

$$F(-x, -y, -z) = -F(x, y, z), \qquad F(0, 0, 0) = 0.$$

The identities are as follows:

$$\text{I}' \quad \begin{aligned} &\sum F(\delta+i,\, \delta-d+i,\, i) = \sum \{F((\Delta_0'+\Delta_0)/2,\, (\Delta_0'-\Delta_0)/2,\, \Delta_0-h) \\ &\qquad -F(-h,\, (\Delta_0'-\Delta_0)/2,\, \Delta_0-h)\} \\ &\qquad +e(n)\sum_{j=1}^{s-1} \{F(j, j, s)-F(s, j, s)\} + a(n)\sum F(r, 0, r-k), \\ &N = i^2+2d\delta = h^2+\Delta_0\Delta_0' = s^2 = r^2+k^2. \end{aligned}$$

$$\text{II}' \quad \begin{aligned} &\sum F(\mu+2d,\, \mu+2d-2\delta,\, \mu) = \sum \{F(\Delta_1'+\Delta_1,\, \Delta_1'-\Delta_1,\, \mu+2\Delta_1) \\ &\qquad -F(\mu,\, \Delta_1'-\Delta_1,\, \mu+2\Delta_1)\} \\ &\qquad +e(n)\sum_{j=1}^{(m-1)/2} \{F(2j-1, 2j-1, m)-F(m, 2j-1, m)\}, \\ &N = \mu^2+8d\delta = \mu^2+4\Delta_1\Delta_1' = m^2. \end{aligned}$$

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* The paraphrased identities can be derived from each other by simpler considerations. For example, to derive II′ from I′ we may let $F(x, y, z)=0$ unless x, y, z are odd; and like manipulations allow other identities to be derived from any one. Such relationships are not unexpected, since these identities are the paraphrases of interdependent theta identities. However, in the interest of simplicity of presentation and of possible applications it is believed better to present them in the form in which they appear.

$$\text{III}'\quad \begin{aligned}&\sum F(t+2i,\, t-\tau+2i,\, 2i)=\sum \{F((t_0+\tau_0)/2,\, (t_0-\tau_0)/2,\, \mu+\tau_0)\\ &\qquad -F(\mu,\, (t_0-\tau_0)/2,\, \mu+\tau_0)\}+a(n)\sum F(m_1,\, 0,\, m_1-m_2),\\ &N=4i^2+2t\tau=\mu^2+t_0\tau_0=m_1^2+m_2^2,\ t,\, \tau \text{ odd}.\end{aligned}$$

$$\text{IV}'\quad \begin{aligned}&\sum F(\mu+t,\, \mu+t-\tau,\, \mu)=\sum \{F((t_2+\tau_2)/2,\, (t_2-\tau_2)/2,\, 2i+\tau_2)\\ &\qquad -F(2i,\, (t_2-\tau_2)/2,\, 2i+\tau_2)\},\\ &N=\mu^2+2t\tau=4i^2+t_2\tau_2,\ t,\, \tau \text{ odd}.\end{aligned}$$

$$\text{V}'\quad \begin{aligned}&\sum F(2t+\mu,\, 2t+\mu-\tau,\, \mu)=\sum F((t_0+\tau_0)/2,\, (t_0-\tau_0)/2,\, 2i+\tau_0)\\ &\qquad -\sum F(\mu,\, \Delta_0'-\Delta_0,\, \mu+2\Delta_0)\\ &\qquad -e(n)\sum_{j=1}^{(m-1)/2} F(m,\, 2j,\, m)+a(n)\sum F(m,\, 0,\, m-2r),\\ &N=\mu^2+4t\tau=4i^2+t_0\tau_0=\mu^2+4\Delta_0\Delta_0'=m^2=m^2+4r^2.\end{aligned}$$

$$\text{VI}'\quad \begin{aligned}&\sum F(2i+2t,\, 2i+2t-\tau,\, 2i)=\sum F((t_2+\tau_2)/2,\, (t_2-\tau_2)/2,\, \mu+\tau_2)\\ &\qquad -\sum F(2i,\, \Delta_1'-\Delta_1,\, 2i+2\Delta_1)-e(n)\sum_{j=1}^{s} F(2s,\, 2j-1,\, 2s),\\ &N=i^2+t\tau=\frac{\mu^2+t_2\tau_2}{4}=i^2+\Delta_1\Delta_1'=s^2.\end{aligned}$$

$$\text{VII}'\quad \begin{aligned}&\sum F(\mu+\tau,\, \mu-2t+\tau,\, \mu)=\sum F(\Delta_0'+\Delta_0,\, \Delta_0'-\Delta_0,\, \mu+2\Delta_0)\\ &\qquad -\sum F(2i,\, (t_0-\tau_0)/2,\, 2i+\tau_0)\\ &\qquad +e(n)\sum_{j=1}^{(m-1)/2} F(2j,\, 2j,\, m)+a(n)\sum F(2r,\, 0,\, 2r-m),\\ &N=\mu^2+4t\tau=\mu^2+4\Delta_0\Delta_0'=4i^2+t_0\tau_0=m^2=m^2+4r^2.\end{aligned}$$

$$\text{VIII}'\quad \begin{aligned}&\sum F(2i+\tau,\, 2i-2t+\tau,\, 2i)=\sum F(\Delta_1'+\Delta_1,\, \Delta_1'-\Delta_1,\, 2h+2\Delta_1)\\ &\qquad -\sum F(\mu,\, (t_2-\tau_2)/2,\, \mu+\tau_2)+e(n)\sum_{j=1}^{s} F(2j-1,\, 2j-1,\, 2s),\\ &N=i^2+t\tau=h^2+\Delta_1\Delta_1'=\frac{\mu^2+t_2\tau_2}{4}=s^2.\end{aligned}$$

If we restrict the parity conditions in I′ by means of the relations

$$F(-x, y, z) = F(x, y, z), \quad F(x, -y, -z) = -F(x, y, z), \quad F(x, 0, z) = 0,$$

and

$$F(-x, y, z) = -F(x, y, z), \quad F(x, -y, -z) = F(x, y, z), \quad F(x, 0, z) = 0,$$

we obtain the two identities of Uspensky mentioned earlier.

4. **The number of representations.** Expressions yielding the number of representations of any number as the sum of three squares in terms of divisor functions can be easily derived from the identities.

We shall make use of the following incomplete numerical functions:

$$g(n) = \sum (-1)^{(d-1)/2}, \qquad n = d\delta,\ 0 < d < \delta,\ d \text{ odd},$$
$$h(n) = \sum (-1)^{(\delta-d-1)/2},\ n = d\delta,\ 0 < d < \delta,\ \delta - d \equiv 1 \pmod 2),$$

and define

$$G(n) = g(n) + 2g(n - 2^2) + 2g(n - 4^2) + \cdots,$$
$$G'(n) = g(n - 1^2) + g(n - 3^2) + g(n - 5^2) + \cdots,$$
$$H(n) = h(n/4) - 2h((n - 4\cdot 1^2)/4) + 2h((n - 4\cdot 2^2)/4) - 2h((n - 4\cdot 3^2)/4) + \cdots.$$

Then, letting

$$F(x, y, z) = (-1)^{x/2+(z-1)/2}, \quad F(x, y, z) = (-1)^{(x-1)/2+z/2},$$
$$F(x, y, z) = (-1)^{x/2-(y-1)/2},$$

in identities IV′ and VII′, in III′, and in VI′, respectively, and letting $N_3(n \equiv k)$ denote the number of representations of $n \equiv k \pmod 8$ as the sum of three squares, we obtain the following theorems:

THEOREM 1. *From* IV′, $N_3(n \equiv 3) = 8G(n)$.

This result is due to Liouville and has been obtained by Uspensky from a special case of identity I′.

THEOREM 2. $N_3(n \equiv 1) = 12G(n) - 12M + 12\sum(-1)^{(m-1)/2}$,* *from* VII′, *where M is* $+1$ *if n is a square of a number of the form* $4k+3$ *and is zero otherwise, and the last sum is over the positive solutions of the equation* $n = m^2 + 4r^2$.

THEOREM 3. *Also from* VII′, $N_3(n \equiv 5) = 12G(n) + 12\sum(-1)^{(m-1)/2}$.

THEOREM 4. *From* III′, $N_3(n \equiv 2) = 24G'(n) + 24K$, *where K is equal to* $+1$ *or* -1 *according as n is the sum of the squares of two numbers both*

* The six possible solutions for a square $x^2 = x^2 + 0 + 0$ are not counted by this formula nor by the starred formulas in Theorems 6 and 7.

of the form $4k+1$ *or both of the form* $4k+3$, *and is zero otherwise.*

THEOREM 5. *From* III$'$, $N_3(n\equiv 6)=24G'(n)$.

THEOREM 6. *From* VI$'$, $N_3(n\equiv 0)=8G'(n)+4H(n)-4e(n)$.*

THEOREM 7. *From* VI$'$, $N_3(n\equiv 4)=8G'(n)+4H(n)-8e(n)$.*

Since $n\equiv 7$ (mod 8) cannot be represented as the sum of three squares, the set of formulas is complete.

It is clear that by selecting other functions $F(x, y, z)$ in a suitable manner other arithmetical results implicit in our general formulas may be obtained. As is usually the case in results of this type, strictly elementary proofs are no doubt possible, but are sometimes difficult to establish even after the theorems are known.

CREIGHTON UNIVERSITY

INVOLUTORY SYSTEMS OF CURVES ON RULED SURFACES†

A. F. CARPENTER

In a paper presented to the International Mathematical Congress in Toronto in 1924, N. B. McLean discussed the properties of a certain one-parameter family of curves lying upon a ruled surface and characterized by the condition of forming a constant cross ratio with the complex curves of the surface. It is the purpose of the present paper to generalize McLean's system of curves and then to call attention to certain interesting special cases.

For the defining system of differential equations we make use of the form

$$\text{(R)}\qquad y'' + p_{12}z' + q_{11}y + q_{12}z = 0, \quad z'' + p_{21}y' + q_{21}y + q_{22}z = 0,$$

where $p_{12}'=2q_{12}$, $p_{21}'=2q_{21}$. For this form the two directrix curves C_y, C_z are the two branches of the flecnode curve of R.

The tetrahedron of reference is that determined by the four points P_y, P_z, P_ρ, P_σ, where

$$\text{(1)}\qquad \rho = 2y' + p_{12}z, \qquad \sigma = 2z' + p_{21}y,$$

the unit point being so chosen that the general point of space will be represented by the expression

$$x_1y + x_2z + x_3\rho + x_4\sigma.$$

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