METRIC SPACES WITH GEODESIC RICCI CURVES. II.

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1. **Introduction.** In this paper we give a partial classification of four-dimensional metric spaces admitting geodesic Ricci curves. The results of I* will be assumed known, along with the notations of that paper.

We assume a given set of independent vectors† $\lambda_{a|}^{i}$ such that $c_{ij}^{i}=0$ (i not summed), so as to obtain geodesic curves, and impose conditions (24), (25), (26) of I on the θ_a. From I, (25) we see that if $\mu_{kk}=0$ for any k, then by I, (24), we have (since $|\lambda_{a|}^{i}|\neq 0$) $\partial\mu_k/\partial x^a=0$, or $\mu_k=\text{const}$. If, however, for any given k, $\mu_{kk}\neq 0$, then from I, (25), $c_{ij}^{k}=0$, for all i and j. This gives us a means of classifying the spaces according to the number of μ_{kk} which equal zero. For $n=4$ there are five cases, which, without loss of generality, we may take in the form:

(A) $\mu_{ii}\neq 0$; (B) $\mu_{11}=0$; $\mu_{22}, \mu_{33}, \mu_{44}\neq 0$;

(C) $\mu_{11}=\mu_{22}=0$; $\mu_{33}, \mu_{44}\neq 0$; (D) $\mu_{11}=\mu_{22}=\mu_{33}=0$; $\mu_{44}\neq 0$;

(E) $\mu_{ii}=0$.

In the following discussion we consider cases (A), (B), and certain special cases under (C). For these special cases we shall merely state the results.

2. **Cases** (A) **and** (B). For case (A) we see from I, (25) that $c_{ij}^{k}=0$, which implies that V_4 is a flat space.

We consider now case (B). Here μ_1 and hence θ_1 is constant. From I, (25) we have

$$\overset{2}{c}_{ij}=\overset{3}{c}_{ij}=\overset{4}{c}_{ij}=0. \tag{1}$$

If in I, (26) we make the substitution

$$\overset{i}{\bar{c}}_{jk}=\frac{\theta_j\theta_k}{\theta_i}\overset{i}{c}_{jk}, \tag{2}$$

we obtain I, (23) in the barred quantities. We call this resulting equation I, (23′).

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* *Metric spaces with geodesic Ricci curves*, I, this Bulletin, vol. 44 (1938), pp. 145–152. We refer to this paper as I, and the notation I, (23), for example, refers to its equation (23).

† All indices take the values 1, 2, 3, 4 unless otherwise noted.

Since by (2), whenever $c^i_{jk}=0$ the corresponding $\bar{c}^i_{jk}$ are also zero, we may solve I, (23′) directly for the $\bar{c}$'s, the normalized form of the c's, without the necessity of first determining the θ's. This process will be made clearer in what follows.

By the use of (1) and (2), I, (23′) reduces to

$$\text{(3)}\qquad \overset{1}{\bar{c}}_{24}\overset{1}{\bar{c}}_{34}=0,\ \overset{1}{\bar{c}}_{23}\overset{1}{\bar{c}}_{43}=0,\ \overset{1}{\bar{c}}_{32}\overset{1}{\bar{c}}_{42}=0,\ \bar{\Delta}_4\overset{1}{\bar{c}}_{34}=0,\ \bar{\Delta}_3\overset{1}{\bar{c}}_{34}=0.$$

We may thus take, for example, $\bar{c}^1_{23}=\bar{c}^1_{24}=0$, and there remains to be determined $\bar{c}^1_{34}$, the only nonzero $\bar{c}$.

A set of operators $\bar{\Delta}_a$ satisfying the relations

$$\text{(4)}\qquad (\bar{\Delta}_a,\bar{\Delta}_b)=\overset{k}{\bar{c}}_{ab}\bar{\Delta}_k,$$

where

$$\bar{\Delta}_a\equiv\bar{\lambda}^i_{a|}\frac{\partial}{\partial x^i},$$

with all $\bar{c}^k_{ab}=0$ except $\bar{c}^1_{34}$, can be expressed in the form

$$\text{(5)}\qquad \bar{\Delta}_\alpha=\frac{\partial}{\partial x^\alpha},\qquad \bar{\Delta}_4=A(x^3,x^4)\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^4},\qquad \alpha=1,2,3;$$

whence $\bar{c}^1_{34}=\partial A/\partial x^3$.

From the last two equations of (3) and (5) we see that $\bar{c}^1_{34}=k$ $=$const. Hence $A=kx^3+G(x^4)$, and by the transformation

$$x'^1=x^1-\int Gdx^4,\qquad x'^\alpha=x^\alpha,\qquad \alpha=2,3,4,$$

we can make $G=0$. As the canonical form for the $\bar{\Delta}$'s we have then (dropping primes)

$$\bar{\Delta}_\alpha=\frac{\partial}{\partial x^\alpha},\qquad \bar{\Delta}_4=kx^3\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^4},\qquad \alpha=1,2,3,$$

where $k\neq 0$.

To obtain the θ_a we use

$$\Delta_j\mu_i=\frac{1}{\theta_j}\bar{\Delta}_j\mu_i=0,\quad\text{or}\quad \bar{\Delta}_j\mu_i=0,\qquad i\neq j.$$

This gives

$$\theta_\alpha=\theta_\alpha(x^\alpha),\quad \alpha=2,3,4;\qquad \theta_1=\text{const}.$$

If we substitute these values for the θ_i in I, (26) and use (2) to obtain c_{34}^1, we will find that I, (26) are satisfied identically. Hence the θ_1, $(i=2, 3, 4)$, are arbitrary functions of their respective arguments. The forms of the $\bar{\lambda}_{a|}^{i}$ and g_{ij} are given in the last section.

3. **Case** (C). In this case θ_1 and θ_2 are constant, and $c_{ij}^3 = c_{ij}^4 = 0$. Hence, by (2),

$$\bar{c}_{ij}^{3} = \bar{c}_{ij}^{4} = 0.$$

It is possible to find a coordinate system in which the $\bar{\Delta}_i$ assume the canonical form

$$(6)\qquad \begin{gathered} \bar{\Delta}_1 = \frac{\partial}{\partial x^1}, \qquad \bar{\Delta}_2 = \frac{\partial}{\partial x^2}, \qquad \bar{\Delta}_3 = \bar{\lambda}_{3|}^{1} \frac{\partial}{\partial x^1} + \bar{\lambda}_{3|}^{2} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \\ \bar{\Delta}_4 = \bar{\lambda}_{4|}^{1} \frac{\partial}{\partial x^1} + \bar{\lambda}_{4|}^{2} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4}, \end{gathered}$$

and from (4) we find

$$(7)\qquad \bar{\lambda}_{3|}^{1}(234), \qquad \bar{\lambda}_{3|}^{2}(134), \qquad \bar{\lambda}_{4|}^{1}(234), \qquad \bar{\lambda}_{4|}^{2}(134),$$

where $f(ij \cdots k)$ means that f is a function of $x^i, x^j, \cdots, x^k$.

For convenience we shall drop the bars on the $\bar{\lambda}$ and $\bar{c}$ and then use the notations

$$\begin{gathered} a = c_{23}^{1}, \quad b = c_{24}^{1}, \quad c = c_{34}^{1}, \quad d = c_{13}^{2}, \quad f = c_{14}^{2}, \quad g = c_{34}^{2}, \\ \alpha = \lambda_{3|}^{1}, \quad \beta = \lambda_{3|}^{2}, \quad \gamma = \lambda_{4|}^{1}, \quad \delta = \lambda_{4|}^{2}. \end{gathered}$$

Then from (4) and (7) we obtain,

$$(8)\qquad \begin{gathered} a = \frac{\partial \alpha}{\partial x^2}, \qquad b = \frac{\partial \gamma}{\partial x^2}, \qquad f = \frac{\partial \delta}{\partial x^1}, \qquad d = \frac{\partial \beta}{\partial x^1}, \\ c = \beta \frac{\partial \gamma}{\partial x^2} - \delta \frac{\partial \alpha}{\partial x^2} + \frac{\partial \gamma}{\partial x^3} - \frac{\partial \alpha}{\partial x^4}, \quad g = \alpha \frac{\partial \delta}{\partial x^1} - \gamma \frac{\partial \beta}{\partial x^1} + \frac{\partial \delta}{\partial x^3} - \frac{\partial \beta}{\partial x^4}; \\ a(234),\ b(234),\ c(1234),\ d(134),\ f(134),\ g(1234),\ \alpha(234), \\ \beta(134),\ \gamma(234),\ \delta(134). \end{gathered}$$

If in I, (23′) we take for the indices bc the values 34, we obtain

$$af + bd + e_1e_2(ab + df) = (f + be_1e_2)(d + ae_1e_2) = 0.$$

Let us take

$$(9)\qquad d = -e_1e_2a,$$

which, from (8), shows that $d(34)$, $a(34)$. The remaining five equations of I, (23′) now have the following form, use being made of (6), (9), and the equation $\Delta_2 a = 0$ which follows from (6):

$$\gamma \frac{\partial c}{\partial x^1} + \delta \frac{\partial c}{\partial x^2} + \frac{\partial c}{\partial x^4} = - e_1 e_2 f g, \tag{10}$$

$$\gamma \frac{\partial g}{\partial x^1} + \delta \frac{\partial g}{\partial x^2} + \frac{\partial g}{\partial x^4} = - e_1 e_2 b c, \tag{11}$$

$$\alpha \frac{\partial c}{\partial x^1} + \beta \frac{\partial c}{\partial x^2} + \frac{\partial c}{\partial x^3} + e_2 e_3 \frac{\partial b}{\partial x^2} = a g, \tag{12}$$

$$\alpha \frac{\partial g}{\partial x^1} + \beta \frac{\partial g}{\partial x^2} + \frac{\partial g}{\partial x^3} + e_1 e_3 \frac{\partial f}{\partial x^1} = - e_1 e_2 a c, \tag{13}$$

$$e_1 \left(\delta \frac{\partial b}{\partial x^2} + \frac{\partial b}{\partial x^4} \right) + e_2 \left(\gamma \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^4} \right) = e_1 e_2 e_3 c g. \tag{14}$$

4. **Subcases of** (C). We shall consider several subcases under (C) and divide these according as $a=0$ and $a \neq 0$:

(C 1) $a = 0$; (C 2) $a \neq 0$.

For (C 1), we have from (9), $d=0$, and by a change of coordinates we can make $\alpha=\beta=0$. If we then differentiate (10) and (11) with respect to x^1 and x^2, we find that

$$f \frac{\partial^2 c}{\partial x^2 \partial x^2} = 0, \qquad b \frac{\partial^2 g}{\partial x^1 \partial x^1} = 0. \tag{15}$$

This leads to the subcases under (C 1):

(C 1.1) $b = f = 0$; (C 1.2) $b \neq 0, f = 0$; (C 1.3) $b \neq 0, f \neq 0$.

The results for these three subcases are as follows:

Case (C 1.1). From (14) we see that c or g is zero. In either case we have case (B) repeated.

Case (C 1.2). For this case we obtain the following two possible solutions:

(16) $b_1 = kB_1'$, $g_1 = B_1$, $c_1 = -\dfrac{e_1 e_2}{k} = c$, $\delta_1 = B_1 x^3$, $\gamma_1 = c x^3 + b_1 x^2$,

(17) $b_2 = kB_2'$, $g_2 = B_2$, $c_2 = c$, $\delta_2 = B_2 x^3$, $\gamma_2 = c x^3 + b_2 x^2$,

where B_1 and B_2 are given by the relations

$$B_1 = k_1 \sin \frac{x^4}{k} + k_2 \cos \frac{x^4}{k}, \quad \text{if} \quad e_1e_3 = +1,$$

$$B_2 = k_1 e^{x^4/k} + k_2 e^{-x^4/k}, \quad \text{if} \quad e_1e_3 = -1, \qquad k,\ k_1,\ k_2 \text{ const.}$$

To obtain the θ_i we proceed as in case (B) and find

$$\theta_1, \theta_2 \text{ const.}, \ \theta_3 = \theta_3(x^3), \ \theta_4 = \theta_4(x^4).$$

If we use (2) to obtain c^1_{24}, c^1_{34}, c^2_{34} (unbarred) and substitute in I, (26), these equations are satisfied identically; hence θ_3 and θ_4 are arbitrary. The forms of $\bar{\lambda}^i_{a|}$ and g_{ij} for this case are given in the last section.

Case (C 1.3). It can be shown that we must have b, c, f, g all functions of x^4 only, and connected by the relations

$$(18) \qquad c' = -e_1e_2fg, \qquad g' = -e_1e_2bc, \qquad e_2b' + e_1f' = e_3cg.$$

If either c or g is zero, then by (18) the other is also, and the third equation of (18) shows that

$$e_2b + e_1f = t = \text{const.},$$

and

$$\gamma = E(x^4)x^2, \qquad \delta = (t - e_2E)e_1x^1,$$

with $E(x^4)$ arbitrary.

As for (C 1.2), $\theta_3(x^3)$, $\theta_4(x^4)$ are arbitrary.

If neither c nor g is zero, we have (18) to determine b, c, f, g, none of which is now zero. By eliminating b and f we obtain

$$(19) \qquad e_1(g'/c)' + e_2(c'/g)' = -e_3cg.$$

We may take, for example, g as arbitrary, and then determine b, c, f from (18) and (19).

Case (C 2). Here $a \neq 0$; whence follows $d \neq 0$. It can be shown that we must have $b \neq 0$, $f \neq 0$ also. We consider only the special case (C 2.1) in which c or g is zero, from which, by (10) or (11), it follows that the other is also. The results for this case, (C 2.1), are:

$$\alpha = a(x^3, x^4)x^2, \quad \beta = -e_1e_2ax^1, \quad \gamma = b(x^3, x^4)x^2, \quad \delta = -e_1e_2bx^4,$$

$$d = -e_1e_2a, \qquad f = -e_1e_2b,$$

and a and b are arbitrary subject to $\partial b/\partial x^3 = \partial a/\partial x^4$.

5. **Forms for the** g_{ij}. In this section we obtain the forms of the g_{ij} for the various cases previously considered. From

$$g^{ij} = \sum_h e_h \bar{\lambda}^i_{h|} \bar{\lambda}^j_{h|}$$

we can easily obtain the g_{ij}. The $\bar{\lambda}^i_{a|}$ are the normalized form of the Ricci congruence vectors and are obtained from $\bar{\Delta}_a = \bar{\lambda}^i_{a|}\partial/\partial x^i$. The canonical forms of the $\bar{\Delta}_a$ have been used for each case. We now use the bars on the λ's and the c's.

Case (A). This gives a flat space, and $\bar{\lambda}^i_{a|} = \delta^i_a$, $g_{ij} = e_i \delta^i_j$.

Case (B). For this case $\bar{c}^1_{34} = k =$ const. (nonzero), and the rest of the $\bar{c}$'s are zero. We have also $\theta_1 = k_1 =$ const., $\theta_\alpha(x^\alpha)$, $(\alpha = 2, 3, 4)$. The $\bar{\lambda}^i_{a|}$ and g_{ij} have, respectively, the following forms:

$$\bar{\lambda}^i_{a|}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ kx^3 & 0 & 0 & 1 \end{pmatrix}, \quad g_{ij}: \begin{pmatrix} e_1 & 0 & 0 & -e_1 k x^3 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ -e_1 k x^3 & 0 & 0 & e_1(kx^3)^2 + e_4 \end{pmatrix}.$$

Case (C). We obtain $\bar{c}^3_{ij} = \bar{c}^4_{ij} = 0$, θ_1, θ_2 const., $\theta_3(x^3)$, $\theta_4(x^4)$.

Case (C 1). The forms of $\bar{\lambda}^i_{a|}$ and g_{ij} are the following:

$$\bar{\lambda}^i_{a|}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma & \delta & 0 & 1 \end{pmatrix}, \quad g_{ij}: \begin{pmatrix} e_1 & 0 & 0 & -e_1\gamma \\ 0 & e_2 & 0 & -e_2\delta \\ 0 & 0 & e_3 & 0 \\ -e_1\gamma & -e_2\delta & 0 & e_1\gamma^2 + e_2\delta^2 + e_4 \end{pmatrix}.$$

Case (C 1.1). This case is equivalent to case (B).

Case (C 1.2). In this case we have $\bar{c}^1_{23} = \bar{c}^2_{13} = \bar{c}^2_{14} = 0$, $\bar{c}^1_{24} \neq 0$,

$$\gamma_1 = cx^3 + (k_1 \cos x^4/k - k_2 \sin x^4/k)x^2,$$

$$\delta_1 = (k_1 \sin x^4/k + k_2 \cos x^4/k)x^3, \qquad e_1 e_3 = +1,$$

$$\gamma_2 = cx^3 + (k_1 e^{x^4/k} - k_2 e^{-x^4/k})x^2, \qquad \delta_2 = (k_1 e^{x^4/k} + k_2 e^{-x^4/k})x^3,$$

$$e_1 e_3 = -1,$$

$$(\bar{c}^1_{24})_1 = k_1 \cos x^4/k - k_2 \sin x^4/k, \qquad (\bar{c}^1_{34})_1 = c = -\frac{e_1 e_2}{k},$$

$$(\bar{c}^2_{34})_1 = k_1 \sin x^4/k + k_2 \cos x^4/k, \qquad (\bar{c}^1_{34})_2 = c = -\frac{e_1 e_2}{k},$$

$$(\bar{c}^1_{24})_2 = k_1 e^{x^4/k} - k_2 e^{-x^4/k}, \quad (\bar{c}^2_{34})_2 = k_1 e^{x^4/k} + k_2 e^{-x^4/k}, \quad k, k_1, k_2 \text{ const.}$$

Case (C 1.3). We have $\bar{c}_{23}^1=\bar{c}_{13}^2=0$, $\bar{c}_{24}^1\neq 0$, $\bar{c}_{14}^2\neq 0$. If $\bar{c}_{34}^1=\bar{c}_{34}^2=0$, then

$$\gamma = E(x^4)x^2, \qquad \delta = (t - e_2E)e_1x^1,$$

$$\bar{b} = \overset{1}{\bar{c}}_{24} = E(x^4), \qquad \bar{f} = \overset{2}{\bar{c}}_{14} = (t - e_2E)e_1, \quad E \text{ arbitrary}, \quad t = \text{const}.$$

If $\bar{c}_{34}^1\bar{c}_{34}^2\neq 0$, then $\bar{c}_{24}^1$, $\bar{c}_{14}^2$, $\bar{c}_{34}^1$, $\bar{c}_{34}^2$ are all functions of x^4 only, connected by equations (18). One of the four functions can be arbitrary, and

$$\gamma = \bar{c}x^3 + \bar{b}x^2, \qquad \delta = \bar{g}x^3 + \bar{f}x^1.$$

Case (C 2). We have $\bar{c}_{23}^1\neq 0$, $\bar{c}_{13}^2\neq 0$, and the form of $\bar{\lambda}_{a|}^i$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \\ \gamma & \delta & 0 & 1 \end{pmatrix}.$$

Case (C 2.1). For this case $\bar{c}_{23}^1$, $\bar{c}_{13}^2$, $\bar{c}_{24}^1$, $\bar{c}_{14}^2\neq 0$, $\bar{c}_{34}^1=\bar{c}_{34}^2=0$, and the g_{ij} is given by

$$\begin{pmatrix} e_1 & 0 & -e_1\alpha & -e_1\gamma \\ 0 & e_2 & -e_2\beta & -e_2\delta \\ -e_1\alpha & -e_2\beta & e_1\alpha^2 + e_2\beta^2 + e_3 & e_1\alpha\gamma + e_2\beta\delta \\ -e_1\gamma & -e_2\delta & e_1\alpha\gamma + e_2\beta\delta & e_1\gamma^2 + e_2\delta^2 + e_4 \end{pmatrix},$$

$\alpha=\bar{a}(x^3, x^4)x^2$, $\beta=-e_1e_2\bar{a}x^1$, $\gamma=\bar{b}(x^3, x^4)x^2$, $\delta=-e_1e_2\bar{b}x^1$, $\bar{a}$ and $\bar{b}$ arbitrary subject to $\partial\bar{b}/\partial x^3=\partial\bar{a}/\partial x^4$.

As stated in I all metric spaces with geodesic Ricci curves for $n=3$ have been obtained. For $n=4$ or greater, this does not seem possible. In this paper we have solved the simplest cases for $n=4$; cases (D) and (E) and the remaining case of (C), in which none of a, b, c, d, f, g are zero, have not been considered.

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