METRIC SPACES WITH GEODESIC RICCI CURVES. II.

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1. Introduction. In this paper we give a partial classification of four-dimensional metric spaces admitting geodesic Ricci curves. The results of I* will be assumed known, along with the notations of that paper.

We assume a given set of independent vectors† $\lambda^i_4$ such that $c^i_0 = 0$ ($i$ not summed), so as to obtain geodesic curves, and impose conditions (24), (25), (26) of I on the $\theta_a$. From I, (25) we see that if $\mu_{kk} = 0$ for any $k$, then by I, (24), we have (since $|\lambda^i_4| \neq 0$) $\partial \mu_k / \partial x^a = 0$, or $\mu_k$ = const. If, however, for any given $k$, $\mu_{kk} \neq 0$, then from I, (25), $c^i_0 = 0$, for all $i$ and $j$. This gives us a means of classifying the spaces according to the number of $\mu_{kk}$ which equal zero. For $n = 4$ there are five cases, which, without loss of generality, we may take in the form:

(A) $\mu_{ii} \neq 0$;  
(B) $\mu_{11} = 0; \mu_{22}, \mu_{33}, \mu_{44} \neq 0$;  
(C) $\mu_{11} = \mu_{22} = 0; \mu_{33}, \mu_{44} \neq 0$;  
(D) $\mu_{11} = \mu_{22} = \mu_{33} = 0; \mu_{44} \neq 0$;  
(E) $\mu_{ii} = 0$.

In the following discussion we consider cases (A), (B), and certain special cases under (C). For these special cases we shall merely state the results.

2. Cases (A) and (B). For case (A) we see from I, (25) that $c^i_0 = 0$, which implies that $V_4$ is a flat space.

We consider now case (B). Here $\mu_1$ and hence $\theta_1$ is constant. From I, (25) we have

\begin{equation}
\frac{\partial}{\partial x^a} \theta_{ij} = \frac{\partial}{\partial x^a} \theta_{ij} = \frac{\partial}{\partial x^a} \theta_{ij} = 0.
\end{equation}

If in I, (26) we make the substitution

\begin{equation}
\tilde{c}_{ik} = \theta_{i} \theta_{k} \tilde{c}_{ik},
\end{equation}

we obtain I, (23) in the barred quantities. We call this resulting equation I, (23').

* Metric spaces with geodesic Ricci curves, I, this Bulletin, vol. 44 (1938), pp. 145-152. We refer to this paper as I, and the notation I, (23), for example, refers to its equation (23).

† All indices take the values 1, 2, 3, 4 unless otherwise noted.
Since by (2), whenever \( c_k = 0 \) the corresponding \( \epsilon_{jk} \) are also zero, we may solve I, (23') directly for the \( \epsilon \)'s, the normalized form of the \( c \)'s, without the necessity of first determining the \( \theta \)'s. This process will be made clearer in what follows.

By the use of (1) and (2), I, (23') reduces to

\[
\begin{align*}
\epsilon_{24}^1 & = 0, \quad \epsilon_{34}^1 = 0, \quad \epsilon_{23}^1 \epsilon_{42}^1 = 0, \quad \Delta_4 c_{34}^1 = 0, \quad \Delta_3 c_{44}^1 = 0. 
\end{align*}
\]

We may thus take, for example, \( \epsilon_{23}^1 = \epsilon_{24}^1 = 0 \), and there remains to be determined \( \epsilon_{34}^1 \), the only nonzero \( \epsilon \).

A set of operators \( \Delta_a \) satisfying the relations

\[
(\Delta_a, \Delta_b) = \epsilon_{ab} \Delta_k,
\]

where

\[
\Delta_a = \lambda_a \frac{\partial}{\partial x^i},
\]

with all \( \epsilon_{ab}^k = 0 \) except \( \epsilon_{34}^1 \), can be expressed in the form

\[
\begin{align*}
\Delta_a & = \frac{\partial}{\partial x^a}, \quad \Delta_4 = A(x^3, x^4) \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad \alpha = 1, 2, 3; \\
\end{align*}
\]

whence \( \epsilon_{34}^1 = \partial A / \partial x^3 \).

From the last two equations of (3) and (5) we see that \( \epsilon_{34}^1 = k \) = const. Hence \( A = k x^3 + G(x^4) \), and by the transformation

\[
x'^1 = x^1 - \int G dx^4, \quad x'^\alpha = x^\alpha, \quad \alpha = 2, 3, 4,
\]

we can make \( G = 0 \). As the canonical form for the \( \Delta \)'s we have then (dropping primes)

\[
\begin{align*}
\Delta_a & = \frac{\partial}{\partial x^a}, \quad \Delta_4 = k x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad \alpha = 1, 2, 3,
\end{align*}
\]

where \( k \neq 0 \).

To obtain the \( \theta_a \) we use

\[
\Delta_{j\mu_i} = \frac{1}{\theta_j} \bar{\Delta}_{j\mu_i} = 0, \quad \text{or} \quad \bar{\Delta}_{j\mu_i} = 0, \quad i \neq j.
\]

This gives

\[
\theta_a = \theta_a(x^\alpha), \quad \alpha = 2, 3, 4; \quad \theta_1 = \text{const}.
\]

If we substitute these values for the \( \theta \) in I, (26) and use (2) to ob-
tain \( c_{34} \), we will find that I, (26) are satisfied identically. Hence the \( \theta_i \), \((i = 2, 3, 4)\), are arbitrary functions of their respective arguments. The forms of the \( \lambda_{4i} \) and \( g_{ij} \) are given in the last section.

3. Case (C). In this case \( \theta_1 \) and \( \theta_2 \) are constant, and \( \epsilon_{ij}^3 = \epsilon_{ij}^4 = 0 \). Hence, by (2),

\[
\dddot{\epsilon}_{ij}^3 = \dddot{\epsilon}_{ij}^4 = 0.
\]

It is possible to find a coordinate system in which the \( \Delta_i \) assume the canonical form

\[
\begin{align*}
\Delta_1 &= \frac{\partial}{\partial x^1}, \\
\Delta_2 &= \frac{\partial}{\partial x^2}, \\
\Delta_3 &= \lambda_{31} \frac{\partial}{\partial x^1} + \lambda_{32} \frac{\partial}{\partial x^2} + \lambda_{33} \frac{\partial}{\partial x^3}, \\
\Delta_4 &= \lambda_{41} \frac{\partial}{\partial x^1} + \lambda_{42} \frac{\partial}{\partial x^2} + \lambda_{43} \frac{\partial}{\partial x^4},
\end{align*}
\]

and from (4) we find

\[
\lambda_{31}^{(234)}, \quad \lambda_{32}^{(134)}, \quad \lambda_{41}^{(234)}, \quad \lambda_{42}^{(134)},
\]

where \( f(ij \cdots k) \) means that \( f \) is a function of \( x^i, x^j, \cdots, x^k \).

For convenience we shall drop the bars on the \( \lambda \) and \( \epsilon \) and then use the notations

\[
a = c_{23}, \quad b = c_{24}, \quad c = c_{34}, \quad d = c_{13}, \quad f = c_{14}, \quad g = c_{34},
\]

\[
\alpha = \lambda_{31}, \quad \beta = \lambda_{32}, \quad \gamma = \lambda_{41}, \quad \delta = \lambda_{42}.
\]

Then from (4) and (7) we obtain,

\[
\begin{align*}
a &= \frac{\partial \alpha}{\partial x^2}, \\
b &= \frac{\partial \gamma}{\partial x^2}, \\
f &= \frac{\partial \delta}{\partial x^3}, \\
d &= \frac{\partial \beta}{\partial x^4},
\end{align*}
\]

\[
\begin{align*}
c &= \beta \frac{\partial \gamma}{\partial x^2} - \delta \frac{\partial \alpha}{\partial x^2} + \gamma \frac{\partial \alpha}{\partial x^3} - \alpha \frac{\partial \alpha}{\partial x^4}, \\
a(234), \quad b(234), \quad c(1234), \quad d(134), \quad f(134), \quad g(1234), \quad \alpha(234), \quad \beta(134), \quad \gamma(234), \quad \delta(134).
\end{align*}
\]

If in I, (23') we take for the indices \( bc \) the values 34, we obtain

\[
a f + b d + e_1 e_2 (a b + d f) = (f + b e_1 e_2)(d + a e_1 e_2) = 0.
\]

Let us take

\[
d = - e_1 e_2 a,
\]

which, from (8), shows that \( d(34), \ a(34) \). The remaining five equa-
tions of I, (23') now have the following form, use being made of (6), (9), and the equation $\Delta g = 0$ which follows from (6):

(10) \[ \gamma \frac{\partial c}{\partial x^1} + \delta \frac{\partial c}{\partial x^2} + \partial \frac{\partial c}{\partial x^4} = -e_1 e_2 fg, \]

(11) \[ \gamma \frac{\partial g}{\partial x^1} + \delta \frac{\partial g}{\partial x^2} + \partial \frac{\partial g}{\partial x^4} = -e_1 e_2 bc, \]

(12) \[ \alpha \frac{\partial c}{\partial x^1} + \beta \frac{\partial c}{\partial x^2} + \partial \frac{\partial c}{\partial x^3} + e_2 e_3 \frac{\partial b}{\partial x^2} = ag, \]

(13) \[ \alpha \frac{\partial g}{\partial x^1} + \beta \frac{\partial g}{\partial x^2} + \partial \frac{\partial g}{\partial x^3} + e_2 e_3 \frac{\partial f}{\partial x^1} = -e_1 e_2 ac, \]

(14) \[ e_1 \left( \delta \frac{\partial b}{\partial x^2} + \frac{\partial b}{\partial x^4} \right) + e_2 \left( \gamma \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^4} \right) = e_1 e_2 e_3 cg. \]

4. Subcases of (C). We shall consider several subcases under (C) and divide these according as $a = 0$ and $a \neq 0$:

(C 1) \[ a = 0; \]

(C 2) \[ a \neq 0. \]

For (C 1), we have from (9), $d = 0$, and by a change of coordinates we can make $\alpha = \beta = 0$. If we then differentiate (10) and (11) with respect to $x^1$ and $x^3$, we find that

(15) \[ f \frac{\partial^2 c}{\partial x^2 \partial x^3} = 0, \quad b \frac{\partial^2 g}{\partial x^2 \partial x^1} = 0. \]

This leads to the subcases under (C 1):

(C 1.1) \[ b = f = 0; \]

(C 1.2) \[ b \neq 0, f = 0; \]

(C 1.3) \[ b \neq 0, f \neq 0. \]

The results for these three subcases are as follows:

Case (C 1.1). From (14) we see that $c$ or $g$ is zero. In either case we have case (B) repeated.

Case (C 1.2). For this case we obtain the following two possible solutions:

(16) \[ b_1 = kB_1', \quad g_1 = B_1, \quad c_1 = -\frac{e_1 e_2}{k} = c, \quad \delta_1 = B_1 x^3, \quad \gamma_1 = c x^3 + b_1 x^2, \]

(17) \[ b_2 = kB_2', \quad g_2 = B_2, \quad c_2 = c, \quad \delta_2 = B_2 x^3, \quad \gamma_2 = c x^3 + b_2 x^2, \]

where $B_1$ and $B_2$ are given by the relations
\[
B_1 = k_1 \sin \frac{x^4}{k} + k_2 \cos \frac{x^4}{k}, \quad \text{if } e_1e_3 = +1,
\]
\[
B_2 = k_1e^{x^4/k} + k_2e^{-x^4/k}, \quad \text{if } e_1e_3 = -1, \quad k, k_1, k_2 \text{ const.}
\]

To obtain the \(\theta_i\) we proceed as in case (B) and find
\[
\theta_1, \theta_2 \text{ const.}, \quad \theta_3 = \theta_3(x^3), \quad \theta_4 = \theta_4(x^4).
\]

If we use (2) to obtain \(c_{24}', c_{34}', c_{34}^a\) (unbarred) and substitute in I, (26), these equations are satisfied identically; hence \(\theta_3\) and \(\theta_4\) are arbitrary. The forms of \(\overline{\lambda}_4^i\) and \(g_{ij}\) for this case are given in the last section.

**Case (C 1.3).** It can be shown that we must have \(b, c, f, g\) all functions of \(x^4\) only, and connected by the relations
\[
\begin{align*}
& c' = -e_1e_2fg, \quad g' = -e_1e_2bc, \quad e_2b' + e_1f' = e_3cg. \\
& \text{If either } c \text{ or } g \text{ is zero, then by (18) the other is also, and the third equation of (18) shows that} \\
& \quad e_2b + e_1f = t = \text{const.},
\end{align*}
\]
and
\[
\gamma = E(x^4)x^2, \quad \delta = (t - e_2E)e_1x^1,
\]
with \(E(x^4)\) arbitrary.

As for (C 1.2), \(\theta_3(x^3), \theta_4(x^4)\) are arbitrary.

If neither \(c\) nor \(g\) is zero, we have (18) to determine \(b, c, f, g\), none of which is now zero. By eliminating \(b\) and \(f\) we obtain
\[
(19) \quad e_1(g'/c)' + e_2(c'/g)' = -e_3cg.
\]

We may take, for example, \(g\) as arbitrary, and then determine \(b, c, f\) from (18) and (19).

**Case (C 2).** Here \(a \neq 0\); whence follows \(d \neq 0\). It can be shown that we must have \(b \neq 0, f \neq 0\) also. We consider only the special case (C 2.1) in which \(c\) or \(g\) is zero, from which, by (10) or (11), it follows that the other is also. The results for this case, (C 2.1), are:
\[
\begin{align*}
& \alpha = a(x^3, x^4)x^2, \quad \beta = -e_1e_2ax^1, \quad \gamma = b(x^3, x^4)x^2, \quad \delta = -e_1e_2bx^4, \\
& d = -e_1e_2a, \quad f = -e_1e_2b,
\end{align*}
\]
and \(a\) and \(b\) are arbitrary subject to \(\partial b/\partial x^3 = \partial a/\partial x^4\).

5. **Forms for the \(g_{ij}\).** In this section we obtain the forms of the \(g_{ij}\) for the various cases previously considered. From
we can easily obtain the $g_{ij}$. The $\lambda_a^i$ are the normalized form of the Ricci congruence vectors and are obtained from $\Delta_a = \lambda_a^i \partial / \partial x^i$. The canonical forms of the $\lambda_a^i$ have been used for each case. We now use the bars on the $\lambda$'s and the $c$'s.

**Case (A).** This gives a flat space, and $\lambda_a^i = \delta_i^i$, $g_{ij} = e_i \delta_j^j$.

**Case (B).** For this case $c_{34}^1 = k = \text{const.}$ (nonzero), and the rest of the $c$'s are zero. We have also $\theta_1 = k_1 = \text{const.}$, $\theta_a(x^a)$, ($\alpha = 2, 3, 4$). The $\lambda_a^i$ and $g_{ij}$ have, respectively, the following forms:

$$
\lambda_a^i: \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
k_x^2 & 0 & 0 & 1
\end{bmatrix},
\ g_{ij}: \begin{bmatrix}
e_1 & 0 & 0 & -e_1 k x^3 \\
0 & e_2 & 0 & 0 \\
0 & 0 & e_3 & 0 \\
-e_1 k x^3 & 0 & 0 & e_1 (k x^2)^2 + e_4
\end{bmatrix}.
$$

**Case (C).** We obtain $c_{ij}^2 = c_{ij}^3 = 0$, $\theta_1$, $\theta_2$ const., $\theta_3(x^2)$, $\theta_4(x^4)$.

**Case (C 1).** The forms of $\lambda_a^i$ and $g_{ij}$ are the following:

$$
\lambda_a^i: \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\gamma & \delta & 0 & 1
\end{bmatrix},
\ g_{ij}: \begin{bmatrix}
e_1 & 0 & 0 & -e_1 \gamma \\
0 & e_2 & 0 & -e_2 \delta \\
0 & 0 & e_3 & 0 \\
-e_1 \gamma & -e_2 \delta & 0 & e_1 \gamma^2 + e_2 \delta^2 + e_4
\end{bmatrix}.
$$

**Case (C 1.1).** This case is equivalent to case (B).

**Case (C 1.2).** In this case we have $c_{23}^1 = c_{13}^2 = c_{14}^2 = 0$, $c_{24}^2 \neq 0$,

$$
\gamma_1 = c x^3 + (k_1 \cos x^4/k - k_2 \sin x^4/k) x^2,
\delta_1 = (k_1 \sin x^4/k + k_2 \cos x^4/k) x^2, \quad e_1 e_3 = +1,
\gamma_2 = c x^3 + (k_1 e^{x^4/k} - k_2 e^{-x^4/k}) x^2, \quad \delta_2 = (k_1 e^{x^4/k} + k_2 e^{-x^4/k}) x^3,
\ e_1 e_3 = -1,
$$

$$
(c_{24}^1)_1 = k_1 \cos x^4/k - k_2 \sin x^4/k, \quad (c_{24}^1)_2 = c = -\frac{e_1 e_2}{k},
(c_{24}^2)_1 = k_1 \sin x^4/k + k_2 \cos x^4/k, \quad (c_{24}^2)_2 = c = -\frac{e_1 e_2}{k},
(c_{24}^1)_2 = k_1 e^{x^4/k} - k_2 e^{-x^4/k}, \quad (c_{24}^2)_2 = k_1 e^{x^4/k} + k_2 e^{-x^4/k}, \quad k_1, k_2 \text{ const.}
$$
Case (C 1.3). We have \( c_{23}^1 = c_{13}^2 = 0, \ c_{24}^1 \neq 0, \ c_{14}^2 \neq 0. \) If \( c_{34}^1 = c_{24}^2 = 0, \) then

\[
\gamma = E(x^4)x^2, \quad \delta = (t - e_2 E)e_1 x^1,
\]

\[
\bar{b} = c_{24}^1 = E(x^4), \quad \bar{f} = c_{14}^2 = (t - e_2 E)e_1, \quad E \text{ arbitrary}, \ t = \text{const}.
\]

If \( c_{34}^1 c_{24}^2 \neq 0, \) then \( c_{23}^1, c_{13}^2, c_{24}^1, c_{34}^2 \) are all functions of \( x^4 \) only, connected by equations (18). One of the four functions can be arbitrary, and

\[
\gamma = \bar{c} x^3 + \bar{b} x^2, \quad \delta = \bar{g} x^3 + \bar{f} x^1.
\]

Case (C 2). We have \( c_{23}^1 \neq 0, c_{13}^2 \neq 0, \) and the form of \( \bar{\lambda}^i_{a1} \) is given by

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\alpha & \beta & 1 & 0 \\
\gamma & \delta & 0 & 1
\end{bmatrix}.
\]

Case (C 2.1). For this case \( c_{23}^1, c_{13}^2, c_{24}^1, c_{14}^2 \neq 0, c_{34}^1 = c_{24}^2 = 0, \) and the \( g_{ij} \) is given by

\[
\begin{pmatrix}
e_1 & 0 & -e_1 \alpha & -e_1 \gamma \\
0 & e_2 & -e_2 \beta & -e_2 \delta \\
-e_1 \alpha & -e_2 \beta & e_1 \alpha^2 + e_2 \beta^2 + e_3 & e_1 \alpha \gamma + e_2 \beta \delta \\
-e_1 \gamma & -e_2 \delta & e_1 \alpha \gamma + e_2 \beta \delta & e_1 \gamma^2 + e_2 \delta^2 + e_4
\end{pmatrix},
\]

\[
\alpha = \bar{a}(x^3, x^4)x^3, \ \beta = -e_1 e_2 \bar{a} x^1, \ \gamma = \bar{b}(x^3, x^4)x^3, \ \delta = -e_1 e_2 \bar{b} x^1, \ \bar{a} \text{ and } \bar{b} \text{ arbitrary subject to } \partial \bar{b}/\partial x^3 = \partial \bar{a}/\partial x^4.
\]

As stated in I all metric spaces with geodesic Ricci curves for \( n = 3 \) have been obtained. For \( n = 4 \) or greater, this does not seem possible. In this paper we have solved the simplest cases for \( n = 4; \) cases (D) and (E) and the remaining case of (C), in which none of \( a, b, c, d, f, g \) are zero, have not been considered.

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