REPRESENTATION OF HOMEOMORPHISMS IN HILBERT SPACE

MORRIS KLINE

1. **Introduction.** Given a normal, topological space S and a continuous transformation f of S into itself, it is possible to imbed S (homeomorphically) in a cartesian space (generally of uncountable dimension) so that the transformation f can be represented as a simple linear transformation of the form

$$y_\alpha = x_\beta, \tag{1}$$

where y_α is the αth coordinate of the transformed point and x_β the βth coordinate of the original point, and α runs through all the coordinate indices. However, the methods of imbedding S whereby such a representation of f becomes possible generally do not permit us to simplify the dimension or structure of the cartesian space when S is more specialized.

Tychonoff has shown* that every normal† topological space S with a neighborhood system of power less than or equal to τ can be imbedded in (a bicompact part of) a cartesian space R_τ in which each point has τ real numbers as coordinates. In particular, if $\tau = \aleph_0$, S can be imbedded in the fundamental parallelopiped of Hilbert space.‡ Now it is possible to refine Tychonoff's method slightly so that a homeomorphism g of S onto itself can be represented by equations of the form (1) acting in R_τ. And when S has a countable neighborhood system, the equations (1) will act in the fundamental parallelopiped of Hilbert space. Thus, to represent a homeomorphism of these more restricted spaces by (1) we can confine ourselves to the fundamental parallelopiped of Hilbert space.

2. **Tychonoff's method.** Tychonoff's method of imbedding a normal topological space S with neighborhood system of power less than or equal to τ in a cartesian space R_τ is as follows.§ We may take as our neighborhood system a basis B for the open sets of S with power

* A. Tychonoff, *Topologische Erweiterung von Räumen*, Mathematische Annalen, vol. 102 (1929), pp. 544–561.

† Complete regularity is sufficient.

‡ By the fundamental parallelopiped of Hilbert space we mean the set of all points $x = (x_1, x_2, \cdots, x_n, \cdots)$ such that $x_n \leq 1/n$ and such that the distance between two points is given by $[\sum_1^\infty (x_i - y_i)^2]^{1/2}$.

§ Loc. cit., pp. 551–552.

τ.* A pair of sets (U, V) of the basis is called canonical if $U \supset \bar{V}$ and if there exists at least one continuous function $f(x)$ on S such that $f=0$ on $\bar{V}, f=1$ on $S-U$, and for every x of S, $0 \leqq f(x) \leqq 1$. There are exactly τ such canonical pairs, and for each one we select one $f(x)$ satisfying these conditions. To these functions we assign distinguishing indices $1, 2, \cdots, \alpha, \cdots$. Then if x is any point of S, we let it correspond to that point x of R_τ which has the coordinates $(f_1(x), f_2(x), \cdots, f_\alpha(x), \cdots)$. We make the following choice of neighborhoods in R_τ. For any ϵ and any choice of a finite number of indices $\alpha_1, \alpha_2, \cdots, \alpha_m$, we understand by a neighborhood of a point x of R_τ all points of R_τ whose α_ith coordinate $(i=1, 2, \cdots, m)$ differs from the α_ith coordinate of x by less than ϵ. With this choice of neighborhoods in R_τ the mapping, just described, of S onto a subset of R_τ is shown by Tychonoff to be a homeomorphism. If S has a countable basis, R_τ is $R_{\aleph_0}$. In this case we make the additional mapping into the fundamental parallelopiped of Hilbert space by means of the homeomorphism

$$x = (x_1, x_2, \cdots, x_n, \cdots), \qquad x' = (x_1, x_2/2, \cdots, x_n/n, \cdots).$$

We note that for the purpose of mapping S into R_τ we can associate to a canonical pair (U, V) any continuous function on S which satisfies the conditions listed above.

3. **Refinement of Tychonoff's method.** Now let S be a normal, topological space, and let g be a homeomorphism of S onto itself.

By an irreducible basis of a space S we mean a basis for the open sets from which no set can be omitted without the loss of the basis property. Any basis gives rise to at least one irreducible basis by omitting as many superfluous sets as necessary.

If the system $\{U\}$ is an irreducible basis for S, then $g(U)$ belongs to this basis. For if not, then $g(U)=U'$ is a sum of sets V_i' each of which belongs to the basis. For each i, $g^{-1}(V_i')=V_i$ is a proper subset of U and is an open set. Moreover, each of these V_i is either a set of the basis or a sum of such sets. Then $U=\sum_i V_i$ is an exact sum of sets of the basis, and the basis is not irreducible.

By the same argument it follows that if U is a set of an irreducible basis, then $g^{-1}(U)$, that is, the pre-image under g of U, is a set of this basis.

If (U, V) is a canonical pair chosen from an irreducible basis, then the pre-images under g of U and V, say U_1 and V_1, form a canonical pair. For we can and do agree to take as a continuous function satis-

* P. Alexandroff and H. Hopf, *Topologie*, vol. 1, p. 42, Theorem VIII.

fying the conditions mentioned in §2 the function which has that value on any point x of S which the function corresponding to the canonical pair (U, V) has on $g(x)$.

Now we regard S as imbedded in the Cartesian space R_τ by the method indicated in §2. It is a simple consequence of the definitions of a homeomorphism and of a real continuous function on a space* that the homeomorphism g of S onto itself can be represented in terms of the coordinates of the Cartesian space in which S is imbedded by means of the equations

$$y_\alpha = g_\alpha(x_1, x_2, \cdots, x_\gamma, \cdots), \tag{2}$$

where α and γ run through all the indices of coordinates in R_τ and each g_α is some real continuous function on the points of S in R_τ.

Because g is a homeomorphism, y_α must take on exactly the set of values of the αth coordinates of points of S. This set of values was determined in the imbedding of S in R_τ by a function f_α corresponding to a canonical pair (U, V). That is, g_α takes on the set of values on S that f_α does. If we take the pre-images under g of U and V, we get a canonical pair (U_1, V_1) such that g_α acts with respect to (U_1, V_1) as f_α acts with respect to (U, V). Because all canonical pairs are used in the imbedding process, and because we have agreed that canonical pairs related as (U, V) and (U_1, V_1) are should have continuous functions attached to them such that the function corresponding to (U_1, V_1), say f_β, has the value on any point x of S that f_α has on $g(x)$, it must be that g_α is f_β. Since f_β determines the x_βth coordinate in R_τ, we may write (2) as

$$y_\alpha = x_\beta, \tag{3}$$

where α runs through all the coordinate indices but the same x_β's may happen to correspond to different y_α's.

If S has a countable basis, the equations (3) are countable in number and represent a transformation in $R_{\aleph_0}$. Then the transformation indicated in §2 which carries S from a subset of $R_{\aleph_0}$ to a subset of the fundamental parallelopiped of Hilbert space does not affect the form of equations (3); hence these equations represent the homeomorphism g in this parallelopiped of Hilbert space.

Institute for Advanced Study

* Cf. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, 1934, p. 25.