GENERALIZED REGULAR RINGS

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1. Introduction. An element $a$ of a ring $\mathcal{R}$ is said to be regular if there exists an element $x$ of $\mathcal{R}$ such that $axa = a$. A ring $\mathcal{R}$ with unit element, every element of which is regular, is a regular ring.† In the present note we introduce rings somewhat more general than the regular rings and prove a few results which are, for the most part, analogous to known theorems about regular rings.‡

Let $\mathcal{R}$ denote a ring with unit element. If for every element $a$ of $\mathcal{R}$ there exists a positive integer $n$ such that $a^n$ is regular, we shall say that $\mathcal{R}$ is $\pi$-regular. In general, the integer $n$ will depend on $a$. If, however, there is a fixed integer $m$ such that for all elements $a$ of $\mathcal{R}$, $a^m$ is regular, we may say that $\mathcal{R}$ is $m$-regular. In this notation, a regular ring is 1-regular.

An important example of a $\pi$-regular ring is a special primary ring, that is, a commutative ring in which every element which is not nilpotent has an inverse.§ It will be seen below that in the study of $\pi$-regular rings the special primary rings play a role similar to that of the fields in the case of regular rings.

2. Theorems on $\pi$-regular rings. Let $\mathcal{R}$ be a $\pi$-regular ring, and $\mathcal{Z}$ its center, that is, the set of all elements commutative with all elements of $\mathcal{R}$. We now prove the first theorem:

**Theorem 1.** The center of a $\pi$-regular ring is $\pi$-regular.

If $a \in \mathcal{Z}$, there exists an $n$ such that for some element $x$ of $\mathcal{R}$, $a^nxa^n = a^n$. Let $y = a^{2n}x^n$. Then, by a trivial modification of von Neumann’s proof of the corresponding result for regular rings,|| it follows that $y$ is in $\mathcal{Z}$ and that $a^nya^n = a^n$. Hence $\mathcal{Z}$ is $\pi$-regular.

* Presented to the Society, September 6, 1938.
‡ In addition to von Neumann, loc. cit., see also a paper by the present author entitled Subrings of infinite direct sums, Duke Mathematical Journal, vol. 4 (1938), pp. 486–494. Hereafter this paper will be referred to as S.
§ See W. Krull, Algebraische Theorie der Ringe, Mathematische Annalen, vol. 88 (1922), pp. 80–122; R. Hölzer, Zur Theorie der primären Ringe, ibid., vol. 96 (1927), pp. 719–735. A ring is primary if every divisor of zero is nilpotent, that is, (0) is a primary ideal.
|| Loc. cit., p. 711.
It is a familiar result* that a ring with unit element is reducible† if and only if its center is reducible. We shall use this fact to establish the following theorem:

**Theorem 2.** A \( \pi \)-regular ring is irreducible if and only if its center is a special primary ring.

In view of the remark just made, we only need to show that the commutative \( \pi \)-regular ring \( \mathfrak{B} \) is irreducible if and only if it is a special primary ring.

It is easy to see that a special primary ring \( \mathfrak{B} \) is irreducible. For if \( \mathfrak{B} \) is the direct sum of two proper ideals, and \( 1 = e_1 + e_2 \) is the corresponding decomposition of the unit, then \( e_i \neq 0 \), \( e_i^2 = e_i \), \((i = 1, 2)\), \( e_1 e_2 = 0 \). Thus \( e_1 \) can be neither nilpotent nor have an inverse, in violation of the definition of a special primary ring.

Suppose now that \( \mathfrak{B} \) is an irreducible commutative \( \pi \)-regular ring, and that \( z \) is any element of \( \mathfrak{B} \) which is not nilpotent. We shall show that \( z \) has an inverse. For some positive integer \( n \), there exists an \( x \) in \( \mathfrak{B} \) such that \( xz^{2n} = z^n \). Now \( xz^n \neq 0 \), as otherwise we should have \( z^n = 0 \). Let \( e_1 = xz^n, e_2 = 1 - e_1 \). Then it is easy to verify that \( e_1^2 = e_1 \), \( e_1 e_2 = 0 \). If \( \mathfrak{B}_1 \) denotes the ideal of all elements of \( \mathfrak{B} \) of the form \( c e_1 \), \( c \in \mathfrak{B} \), \((i = 1, 2)\), then \( \mathfrak{B} \) is the direct sum of the ideals \( \mathfrak{B}_1 \) and \( \mathfrak{B}_2 \). Since \( \mathfrak{B}_1 \neq 0 \), our assumption that \( \mathfrak{B} \) is irreducible requires that \( \mathfrak{B}_2 = 0 \). Thus \( e_2 = 0 \), which implies that \( z \) has the inverse \( xz^{n-1} \).

We now prove the following theorem:

**Theorem 3.** In a commutative \( \pi \)-regular ring \( \mathfrak{R} \), every prime ideal is divisorless.

Let \( \mathfrak{p} \) be an arbitrary prime ideal in \( \mathfrak{R} \). Then the ring \( \mathfrak{R}/\mathfrak{p} \) contains no divisors of zero and hence is irreducible. But clearly \( \mathfrak{R}/\mathfrak{p} \) is a commutative \( \pi \)-regular ring, and hence by the preceding theorem must be a special primary ring. However a special primary ring without divisors of zero is a field, and this implies that \( \mathfrak{p} \) is divisorless.‡

The final theorem of this section now follows immediately from a theorem of Krull.§

**Theorem 4.** In a commutative \( \pi \)-regular ring every ideal is the intersection of its primary ideal divisors.

† That is, expressible as a direct sum of two proper two-sided ideals.
‡ Cf. S, Theorem 8.
3. Characterizations of commutative $\pi$-regular and $m$-regular rings. From the preceding theorem it follows* that a commutative $\pi$-regular ring is isomorphic to a subring of a direct sum of primary rings, there being in general an infinite number of summands. But a primary ring can be imbedded in a special primary ring,† and we thus have the theorem:

THEOREM 5. A commutative $\pi$-regular ring is isomorphic to a subring of a direct sum of special primary rings.

In any commutative ring, if a primary ideal $q$ has the property that whenever a finite power of an element $b$ is in $q$, then $b^m \equiv 0 \pmod{q}$, we shall say that $q$ is a primary ideal of index $m$. In other words, the primary ideal $q$ has index $m$ if and only if $x^m = 0$ for every element $x$ in the radical of $R/q$. It is obvious that a primary ideal of index $m$ is also primary of index $k$, where $k$ is any positive integer greater than $m$. A prime ideal is clearly a primary ideal of index 1. We may remark also that if a commutative ring is $m$-regular it is also $(m+1)$-regular and therefore $k$-regular if $k > m$. For if $a^{2m}x = a^m$, it is easily verified that

$$a^{2(m+1)}(a^{2m-1}x^2) = a^{m+1},$$

and this implies that $a^{m+1}$ is regular.

It is now easy to establish the following generalization of a known theorem on regular rings:**

THEOREM 6. A necessary and sufficient condition that a commutative ring $R$, with unit element, be $m$-regular is that in $R$ every ideal be the intersection of its primary ideal divisors of index $m$.

If $R$ is $m$-regular, then every primary ideal is of index $m$. For if $q$ is a primary ideal and $a^k \equiv 0 \pmod{q}$, $(k > m)$, then since $a^{2m}x = a^m$, it follows that for each positive integer $i > 1$,

$$a^{im}x = a^{(i-1)m}.$$  

But for some $i$, $a^{im} \equiv 0 \pmod{q}$, and thus $a^{(i-1)m} \equiv 0 \pmod{q}$. A repetition finally shows that $a^m \equiv 0 \pmod{q}$. Hence $q$ is of index $m$, and Theorem 4 completes the proof of the first part of the theorem.

Conversely, suppose $R$ is a commutative ring with unit element in which every ideal is the intersection of its primary divisors of index $m$. Let $a$ be an arbitrary element of $R$. We shall show that there exists

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* S, Theorem 1.
† See Hölzer, loc. cit., p. 722.
‡ S, Theorem 9.
an \( x \) such that \( a^{2m}x = a^m \). Let \( q \) denote an arbitrary primary divisor of \( (a^{2m}) \) of index \( m \). Then also \( a^m = 0 \) (\( q \)) as follows at once from the assumption that \( q \) is of index \( m \). Hence \( (a^m) \) and \( (a^{2m}) \) have precisely the same primary ideal divisors of index \( m \); thus, by hypothesis, it follows that \( (a^m) = (a^{2m}) \). That is, there exists an \( x \) such that \( a^{2m}x = a^m \), and \( a^m \) is regular. Thus \( \mathfrak{R} \) is \( m \)-regular.

We conclude with the following theorem:

**Theorem 7.** A necessary and sufficient condition that a commutative ring \( \mathfrak{R} \), with unit element, be \( m \)-regular is that all direct indecomposable ideals be primary of index \( m \).*

It is known† that in an arbitrary ring with unit element every ideal is the intersection of its direct indecomposable ideal divisors. If these are all primary of index \( m \), the preceding theorem shows that \( \mathfrak{R} \) is \( m \)-regular.

Suppose \( \mathfrak{R} \) is \( m \)-regular, and let \( \mathfrak{f} \) be a direct indecomposable ideal in \( \mathfrak{R} \). Then \( \mathfrak{R}/\mathfrak{f} \) is irreducible and is also \( m \)-regular. Thus, by Theorem 2, \( \mathfrak{R}/\mathfrak{f} \) is a special primary ring and \( \mathfrak{f} \) is therefore a primary ideal in \( \mathfrak{R} \). Theorem 6 then states that \( \mathfrak{f} \) is of index \( m \), and the proof is completed.

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**A FORMULA FOR THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL‡**

**J. E. Eaton**

Despite the widespread use of the roots of unity in the solution of many mathematical questions, the problem of characterizing the irreducible equation

\[
F_n(x) = x^r + a_1x^{r-1} + \cdots + a_r = 0
\]

whose roots are the primitive \( n \)th roots of unity has received little attention. It is well known that \( r = \phi(n) \), that \( F_n(1) = p \) for \( n = p^s \) (where \( p \) is a prime) and \( F_n(1) = 1 \) otherwise. For \( n \) a power of a prime \( a_i \) is 1 or 0. In 1883 Migotti§ proved that for \( n \) a product of two primes \( a_i \) is \( \pm 1 \) or 0. In 1895 Bang‖ showed that for \( n \) a product of

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* Cf. S, Theorem 10.
† See S, §4.
‡ Presented to the Society, February 26, 1938.
§ Sitzungsberichte der Akademie der Wissenschaften, Vienna, (2), vol. 87 (1883), pp. 7–14.
‖ Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.