GENERALIZED REGULAR RINGS*

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1. **Introduction.** An element a of a ring $\mathfrak{R}$ is said to be *regular* if there exists an element x of $\mathfrak{R}$ such that $axa=a$. A ring $\mathfrak{R}$ with unit element, every element of which is regular, is a *regular ring*.† In the present note we introduce rings somewhat more general than the regular rings and prove a few results which are, for the most part, analogous to known theorems about regular rings.‡

Let $\mathfrak{R}$ denote a ring with unit element. If for every element a of $\mathfrak{R}$ there exists a positive integer n such that a^n is regular, we shall say that $\mathfrak{R}$ is *π-regular*. In general, the integer n will depend on a. If, however, there is a fixed integer m such that for all elements a of $\mathfrak{R}$, a^m is regular, we may say that $\mathfrak{R}$ is *m-regular*. In this notation, a regular ring is 1-regular.

An important example of a π-regular ring is a special primary ring, that is, a commutative ring in which every element which is not nilpotent has an inverse.§ It will be seen below that in the study of π-regular rings the special primary rings play a role similar to that of the fields in the case of regular rings.

2. **Theorems on π-regular rings.** Let $\mathfrak{R}$ be a π-regular ring, and $\mathfrak{Z}$ its center, that is, the set of all elements commutative with all elements of $\mathfrak{R}$. We now prove the first theorem:

THEOREM 1. *The center of a π-regular ring is π-regular.*

If $a \in \mathfrak{Z}$, there exists an n such that for some element x of $\mathfrak{R}$, $a^nxa^n=a^n$. Let $y=a^{2n}x^3$. Then, by a trivial modification of von Neumann's proof of the corresponding result for regular rings,‖ it follows that y is in $\mathfrak{Z}$ and that $a^nya^n=a^n$. Hence $\mathfrak{Z}$ is π-regular.

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† J. von Neumann, *On regular rings*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707–713.

‡ In addition to von Neumann, loc. cit., see also a paper by the present author entitled *Subrings of infinite direct sums*, Duke Mathematical Journal, vol. 4 (1938), pp. 486–494. Hereafter this paper will be referred to as S.

§ See W. Krull, *Algebraische Theorie der Ringe*, Mathematische Annalen, vol. 88 (1922), pp. 80–122; R. Hölzer, *Zur Theorie der primären Ringe*, ibid., vol. 96 (1927), pp. 719–735. A ring is *primary* if every divisor of zero is nilpotent, that is, (0) is a primary ideal.

‖ Loc. cit., p. 711.

It is a familiar result* that a ring with unit element is reducible† if and only if its center is reducible. We shall use this fact to establish the following theorem:

THEOREM 2. *A π-regular ring is irreducible if and only if its center is a special primary ring.*

In view of the remark just made, we only need to show that the commutative π-regular ring $\mathfrak{Z}$ is irreducible if and only if it is a special primary ring.

It is easy to see that a special primary ring $\mathfrak{Z}$ is irreducible. For if $\mathfrak{Z}$ is the direct sum of two proper ideals, and $1=e_1+e_2$ is the corresponding decomposition of the unit, then $e_i\neq 0$, $e_i^2=e_i$, $(i=1, 2)$, $e_1e_2=0$. Thus e_1 can be neither nilpotent nor have an inverse, in violation of the definition of a special primary ring.

Suppose now that $\mathfrak{Z}$ is an irreducible commutative π-regular ring, and that z is any element of $\mathfrak{Z}$ which is not nilpotent. We shall show that z has an inverse. For some positive integer n, there exists an x in $\mathfrak{Z}$ such that $xz^{2n}=z^n$. Now $xz^n\neq 0$, as otherwise we should have $z^n=0$. Let $e_1=xz^n$, $e_2=1-e_1$. Then it is easy to verify that $e_i^2=e_i$, $e_1e_2=0$. If $\mathfrak{Z}_i$ denotes the ideal of all elements of $\mathfrak{Z}$ of the form ce_i, $c \in \mathfrak{Z}$, $(i=1, 2)$, then $\mathfrak{Z}$ is the direct sum of the ideals $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$. Since $\mathfrak{Z}_1\neq 0$, our assumption that $\mathfrak{Z}$ is irreducible requires that $\mathfrak{Z}_2=0$. Thus $e_2=0$, which implies that z has the inverse xz^{n-1}.

We now prove the following theorem:

THEOREM 3. *In a commutative π-regular ring $\mathfrak{R}$, every prime ideal is divisorless.*

Let $\mathfrak{p}$ be an arbitrary prime ideal in $\mathfrak{R}$. Then the ring $\mathfrak{R}/\mathfrak{p}$ contains no divisors of zero and hence is irreducible. But clearly $\mathfrak{R}/\mathfrak{p}$ is a commutative π-regular ring, and hence by the preceding theorem must be a special primary ring. However a special primary ring without divisors of zero is a field, and this implies that $\mathfrak{p}$ is divisorless.‡

The final theorem of this section now follows immediately from a theorem of Krull.§

THEOREM 4. *In a commutative π-regular ring every ideal is the intersection of its primary ideal divisors.*

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* Cf. van der Waerden, *Moderne Algebra*, vol. 2, p. 164.

† That is, expressible as a direct sum of two proper two-sided ideals.

‡ Cf. S, Theorem 8.

§ W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Mathematische Annalen, vol. 101 (1929), p. 738.

3. **Characterizations of commutative π-regular and m-regular rings.** From the preceding theorem it follows* that a commutative π-regular ring is isomorphic to a subring of a direct sum of primary rings, there being in general an infinite number of summands. But a primary ring can be imbedded in a special primary ring,† and we thus have the theorem:

THEOREM 5. *A commutative π-regular ring is isomorphic to a subring of a direct sum of special primary rings.*

In any commutative ring, if a primary ideal $\mathfrak{q}$ has the property that whenever a finite power of an element b is in $\mathfrak{q}$, then $b^m \equiv 0$ ($\mathfrak{q}$), we shall say that $\mathfrak{q}$ is a primary ideal of *index m*. In other words, the primary ideal $\mathfrak{q}$ has index m if and only if $x^m = 0$ for every element x in the radical of $\mathfrak{R}/\mathfrak{q}$. It is obvious that a primary ideal of index m is also primary of index k, where k is any positive integer greater than m. A prime ideal is clearly a primary ideal of index 1. We may remark also that if a commutative ring is m-regular it is also $(m+1)$-regular and therefore k-regular if $k > m$. For if $a^{2m}x = a^m$, it is easily verified that

$$a^{2(m+1)}(a^{2m-1}x^3) = a^{m+1},$$

and this implies that a^{m+1} is regular.

It is now easy to establish the following generalization of a known theorem on regular rings:‡

THEOREM 6. *A necessary and sufficient condition that a commutative ring $\mathfrak{R}$, with unit element, be m-regular is that in $\mathfrak{R}$ every ideal be the intersection of its primary ideal divisors of index m.*

If $\mathfrak{R}$ is m-regular, then every primary ideal is of index m. For if $\mathfrak{q}$ is a primary ideal and $a^k \equiv 0$ ($\mathfrak{q}$), $(k > m)$, then since $a^{2m}x = a^m$, it follows that for each positive integer $i > 1$,

$$a^{im}x = a^{(i-1)m}.$$

But for some i, $a^{im} \equiv 0$ ($\mathfrak{q}$), and thus $a^{(i-1)m} \equiv 0$ ($\mathfrak{q}$). A repetition finally shows that $a^m \equiv 0$ ($\mathfrak{q}$). Hence $\mathfrak{q}$ is of index m, and Theorem 4 completes the proof of the first part of the theorem.

Conversely, suppose $\mathfrak{R}$ is a commutative ring with unit element in which every ideal is the intersection of its primary divisors of index m. Let a be an arbitrary element of $\mathfrak{R}$. We shall show that there exists

* S, Theorem 1.

† See Hölzer, loc. cit., p. 722.

‡ S, Theorem 9.

an x such that $a^{2m}x=a^m$. Let $\mathfrak{q}$ denote an arbitrary primary divisor of (a^{2m}) of index m. Then also $a^m\equiv 0$ $(\mathfrak{q})$ as follows at once from the assumption that $\mathfrak{q}$ is of index m. Hence (a^m) and (a^{2m}) have precisely the same primary ideal divisors of index m; thus, by hypothesis, it follows that $(a^m)=(a^{2m})$. That is, there exists an x such that $a^{2m}x=a^m$, and a^m is regular. Thus $\mathfrak{R}$ is m-regular.

We conclude with the following theorem:

THEOREM 7. *A necessary and sufficient condition that a commutative ring $\mathfrak{R}$, with unit element, be m-regular is that all direct indecomposable ideals be primary of index m.**

It is known† that in an arbitrary ring with unit element every ideal is the intersection of its direct indecomposable ideal divisors. If these are all primary of index m, the preceding theorem shows that $\mathfrak{R}$ is m-regular.

Suppose $\mathfrak{R}$ is m-regular, and let $\mathfrak{k}$ be a direct indecomposable ideal in $\mathfrak{R}$. Then $\mathfrak{R}/\mathfrak{k}$ is irreducible and is also m-regular. Thus, by Theorem 2, $\mathfrak{R}/\mathfrak{k}$ is a special primary ring and $\mathfrak{k}$ is therefore a primary ideal in $\mathfrak{R}$. Theorem 6 then states that $\mathfrak{k}$ is of index m, and the proof is completed.

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A FORMULA FOR THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL‡

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Despite the widespread use of the roots of unity in the solution of many mathematical questions, the problem of characterizing the irreducible equation

$$F_n(x) = x^r + a_1x^{r-1} + \cdots + a_r = 0$$

whose roots are the primitive nth roots of unity has received little attention. It is well known that $r=\phi(n)$, that $F_n(1)=p$ for $n=p^\alpha$ (where p is a prime) and $F_n(1)=1$ otherwise. For n a power of a prime a_i is 1 or 0. In 1883 Migotti§ proved that for n a product of two primes a_i is ± 1 or 0. In 1895 Bang|| showed that for n a product of

* Cf. S, Theorem 10.

† See S, §4.

‡ Presented to the Society, February 26, 1938.

§ Sitzungsberichte der Akademie der Wissenschaften, Vienna, (2), vol. 87 (1883), pp. 7–14.

|| Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.