JACKSON SUMMATION OF THE FABER DEVELOPMENT*

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1. Introduction. The purpose of this note is to prove the following theorem:

**THEOREM.** Let \( C \) be an analytic Jordan curve in the \( z \)-plane, and let \( f(z) \) be analytic in \( C \), continuous in \( C \), the closed limited set bounded by \( C \), and let \( f^{(p)}(z) \), \((p \geq 0)\), satisfy a Lipschitz condition\(^\dagger\) of order \( \alpha \), \((0 < \alpha \leq 1)\), on \( C \). Then

\[
\left| f(z) - \sum_{n=0}^{n} a_n P_n(z) \right| \leq \frac{M}{n^{p+\alpha}}, \quad z \text{ in } C,
\]

where \( M \) is a constant independent of \( n \) and \( z \),

\[
\sum_{n=0}^{n} a_n P_n(z)
\]

is the sum of the first \( n+1 \) terms of the development of \( f(z) \) in the Faber\(^\S\) polynomials belonging to \( C \), and \( d_n \) is the Jackson\(^\|$ summation coefficient of order \( p \).

In a previous paper the author\(^\|$ showed that under the above hypothesis

\[
\left| f(z) - \sum_{0}^{n} a_r P_r(z) \right| \leq M (\log n/n^{p+\alpha}) , \quad z \text{ in } C.
\]

Later John Curtiss\(^\|\) proved the existence of a sequence of polynomials \( Q_n(z) \) of respective degrees \( n \), \((n = 1, 2, \cdots)\), such that

\[
\left| f(z) - Q_n(z) \right| \leq M/n^{p+\alpha}.
\]

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† \( f^{(p)}(z) \) denotes the \( p \)th derivative of \( f(z) \); \( f^{(0)}(z) = f(z) \).

‡ \( f(z) \) satisfies a Lipschitz condition of order \( \alpha \) on \( C \) if for \( z_1 \) and \( z_2 \) arbitrary points on \( C \) we have \( |f(z_1) - f(z_2)| \leq L |z_1 - z_2|^\alpha \), where \( L \) is a constant independent of \( z_1 \) and \( z_2 \).

§ G. Faber, Mathematische Annalen, vol. 57 (1903), pp. 389–408.


‖ This Bulletin, vol. 41 (1935), pp. 111–117; this paper will be referred to hereafter as SI.


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In a paper soon to appear, Walsh and the author state that for \( p = 0 \), \((0 < \alpha < 1)\), the inequality
\[
|f(z) - \sigma_n(z)| \leq M/\alpha^n
\]
is valid, where \( \sigma_n(z) \) is the \( n \)th arithmetic mean of the development of \( f(z) \) in the Faber polynomials belonging to \( C \). Here we extend this result by exhibiting a set of polynomials (proved by Curtiss to exist) with the prescribed degree of convergence for arbitrary \( p \) and for \( 0 < \alpha \leq 1 \).

2. Proof of the theorem. Let

\[
z = \psi(t) = \frac{1}{t} + b_0 + b_1 t + b_2 t^2 + \cdots = \frac{1}{t} + \Psi(t)
\]

map the exterior of \( C \) on the region \(|t| < r\) of the complex \( t \)-plane so that the point \( z = \infty \) corresponds to the point \( t = 0 \). It follows from the analyticity of \( C \) that the right-hand side of \((2)\) converges for \(|t| \leq r', \ r' > r \). The Faber polynomials belonging to \( C \) are defined as follows: \( P_n(z) \) is the polynomial of degree \( n \) in \( z \) such that the coefficient of \( z^n \) is unity, and, as a function of \( t \) through \((2)\), such that the coefficients are zero for the terms in \( t^{n+1}, t^{n+2}, \ldots, t^0 \); hence \( P_n(z) = 1/t^n + t \Psi_n(t) \), where \( \Psi_n(t) \) converges for \(|t| \leq r' \).

Faber (loc. cit.) and the author (SI) have shown that
\[
f(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad z \text{ in } C,
\]

\[
= \sum_{n=0}^{\infty} a_n \left( \frac{1}{t^n} + t \Psi_n(t) \right) = \sum_{n=0}^{\infty} a_n \frac{1}{t^n} + \sum_{n=0}^{\infty} c_n t^n, \quad r \leq |t| \leq r'.
\]

It should be noted here that (Faber, loc. cit.) \(|a_n| \leq M_1 r^n \) and \(|\Psi_n(t)| \leq M_2/\alpha^n r^n \), where \( M_1 \) and \( M_2 \) are constants.

Now we consider

\[
f(z) - \sum_{n=0}^{n} d_n a_n P_n(z) = f(\psi(t)) - \sum_{n=0}^{n} d_n a_n \left( \frac{1}{t^n} + t \Psi_n(t) \right)
\]

\[
= f(\psi(t)) - \sum_{n=1}^{\infty} c_n t^n - \sum_{n=0}^{n} d_n a_n /t^n
\]

\[
- \sum_{n=1}^{n} d_n a_n \Psi_n(t) + \sum_{n=1}^{\infty} c_n t^n
\]

\[
= f(\psi(t)) - \sum_{n=1}^{\infty} c_n t^n - \sum_{n=0}^{n} d_n a_n /t^n
\]

\[
+ \sum_{n=1}^{n} (1 - d_n) a_n \Psi_n(t) + \sum_{n+1}^{\infty} a_n \Psi_n(t), \quad r \leq |t| \leq r'.
\]
The Faber Development

Therefore

\[ |f(x) - \sum_{r=0}^{n} d_{nr} a_r \Psi_r(x)| \leq |f(\psi(t)) - \sum_{r=1}^{\infty} c_r t^r - \sum_{r=0}^{n} d_{nr} a_r / t^r| + \left| \sum_{r=1}^{n} (1 - d_{nr}) a_r \Psi_r(t) \right| + \left| \sum_{r=1}^{n+1} a_r \Psi_r(t) \right|, \quad r \leq |t| \leq r'. \]

The sum \( \sum_{r=0}^{n} d_{nr} a_r / t^r \) is (SI) the Jackson summation of the first \( n+1 \) terms of the Taylor development of the function \( f(\psi(t)) - \sum_{r=1}^{\infty} c_r t^r \) and consequently (Jackson, loc. cit.; Curtiss, loc. cit.)

\[ |f(\psi(t)) - \sum_{r=1}^{\infty} c_r t^r - \sum_{r=0}^{n} d_{nr} a_r / t^r| \leq \frac{M_3}{n^{p+\alpha}}, \quad r \leq |t| \leq r'. \]

Also

\[ \left| \sum_{n+1}^{\infty} a_r \Psi_r(t) \right| \leq \sum_{n+1}^{\infty} |a_r| |t| |\Psi_r(t)| \leq M_1 M_\eta \sum_{n+1}^{\infty} \left( \frac{r}{r'} \right)^p \]

\[ \leq \frac{M_4}{n^{p+\alpha}}, \quad |t| = r, \]

since \( r/r' < 1 \). Furthermore*

\[ |1 - d_{nr}| \leq \frac{M_4 r^{p+1}}{n^{p+1}}, \]

consequently

\[ \left| \sum_{r=1}^{n} (1 - d_{nr}) a_r \Psi_r(t) \right| \leq M_1 M_2 M_5 \sum_{1}^{n} \frac{r^{p+1}}{n^{p+1}} \left( \frac{r}{r'} \right)^p = \frac{M_1 M_2 M_5}{n^{p+1}} \sum_{1}^{n} r^{p+1} \left( \frac{r}{r'} \right)^p. \]

The series on the right converges, and the proof of the theorem is thus complete.

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* Dunham Jackson, Transactions of this Society, vol. 15 (1914), pp. 439-466. It should be noted here that Jackson’s summation coefficients vary with the derivative but not with the order of the Lipschitz condition.