JACKSON SUMMATION OF THE FABER DEVELOPMENT*

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1. **Introduction.** The purpose of this note is to prove the following theorem:

THEOREM. *Let C be an analytic Jordan curve in the z-plane, and let $f(z)$ be analytic in C, continuous in $\overline{C}$, the closed limited set bounded by C, and let*† $f^{(p)}(z)$, $(p \geqq 0)$, *satisfy a Lipschitz condition*‡ *of order α, $(0<\alpha \leqq 1)$, on C. Then*

$$(1) \qquad \left| f(z) - \sum_{\nu=0}^{n} d_{n\nu} a_\nu P_\nu(z) \right| \leqq \frac{M}{n^{p+\alpha}}, \qquad z \text{ in } \overline{C},$$

where M is a constant independent of n and z,

$$\sum_{\nu=0}^{n} a_\nu P_\nu(z)$$

is the sum of the first $n+1$ terms of the development of $f(z)$ in the Faber§ *polynomials belonging to C, and $d_{n\nu}$ is the Jackson*|| *summation coefficient of order p.*

In a previous paper the author¶ showed that under the above hypothesis

$$\left| f(z) - \sum_{0}^{n} a_\nu P_\nu(z) \right| \leqq M(\log n/n^{p+\alpha}), \qquad z \text{ in } \overline{C}.$$

Later John Curtiss** proved the existence of a sequence of polynomials $Q_n(z)$ of respective degrees n, $(n=1, 2, \cdots)$, such that

$$|f(z) - Q_n(z)| \leqq M/n^{p+\alpha}.$$

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† $f^{(p)}(z)$ denotes the pth derivative of $f(z)$; $f^{(0)}(z) \equiv f(z)$.

‡ $f(z)$ satisfies a Lipschitz condition of order α on C if for z_1 and z_2 arbitrary points on C we have $|f(z_1)-f(z_2)| \leqq L|z_1-z_2|^\alpha$, where L is a constant independent of z_1 and z_2.

§ G. Faber, Mathematische Annalen, vol. 57 (1903), pp. 389–408.

|| Dunham Jackson, Transactions of this Society, vol. 15 (1914), pp. 439–466; p. 463.

¶ This Bulletin, vol. 41 (1935), pp. 111–117; this paper will be referred to hereafter as SI.

** This Bulletin, vol. 42 (1936), pp. 873–878.

In a paper soon to appear, Walsh and the author state that for $p=0$, $(0<\alpha<1)$, the inequality $|f(z)-\sigma_n(z)|\leqq M/n^\alpha$ is valid, where $\sigma_n(z)$ is the nth arithmetic mean of the development of $f(z)$ in the Faber polynomials belonging to C. Here we extend this result by exhibiting a set of polynomials (proved by Curtiss to exist) with the prescribed degree of convergence for arbitrary p and for $0<\alpha\leqq 1$.

2. **Proof of the theorem.** Let

$$z = \psi(t) = \frac{1}{t} + b_0 + b_1 t + b_2 t^2 + \cdots = \frac{1}{t} + \mathfrak{P}(t) \tag{2}$$

map the exterior of C on the region $|t|<r$ of the complex t-plane so that the point $z=\infty$ corresponds to the point $t=0$. It follows from the analyticity of C that the right-hand side of (2) converges for $|t|\leqq r'$, $r'>r$. The Faber polynomials belonging to C are defined as follows: $P_n(z)$ is the polynomial of degree n in z such that the coefficient of z^n is unity, and, as a function of t through (2), such that the coefficients are zero for the terms in $t^{-n+1}, t^{-n+2}, \cdots, t^{-1}, t^0$; hence $P_n(z)=1/t^n+t\mathfrak{P}_n(t)$, where $\mathfrak{P}_n(t)$ converges for $|t|\leqq r'$.

Faber (loc. cit.) and the author (SI) have shown that

$$\begin{aligned} f(z) &= \sum_{\nu=0}^{\infty} a_\nu P_\nu(z), && z \text{ in } \overline{C}, \\ &= \sum_{\nu=0}^{\infty} a_\nu\left(\frac{1}{t^\nu} + t\mathfrak{P}_\nu(t)\right) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{t^\nu} + \sum_{1}^{\infty} c_\nu t^\nu, && r \leqq |t| \leqq r'. \end{aligned}$$

It should be noted here that (Faber, loc. cit.) $|a_\nu|\leqq M_1 r^\nu$ and $|\mathfrak{P}_\nu(t)|\leqq M_2/r'^\nu$, where M_1 and M_2 are constants.

Now we consider

$$\begin{aligned} f(z) - \sum_{\nu=0}^{n} d_{n\nu}a_\nu P_\nu(z) &= f(\psi(t)) - \sum_{\nu=0}^{n} d_{n\nu}a_\nu\left(\frac{1}{t^\nu} + t\mathfrak{P}_\nu(t)\right) \\ &= f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu}a_\nu/t^\nu \\ &\quad - \sum_{\nu=1}^{n} d_{n\nu}a_\nu t\mathfrak{P}_\nu(t) + \sum_{\nu=1}^{\infty} c_\nu t^\nu \\ &= f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu}a_\nu/t^\nu \\ &\quad + \sum_{\nu=1}^{n} (1-d_{n\nu})a_\nu t\mathfrak{P}_\nu(t) + \sum_{n+1}^{\infty} a_\nu t\mathfrak{P}_\nu(t), \\ &\qquad\qquad r \leqq |t| \leqq r'. \end{aligned}$$

Therefore

$$\left|f(z) - \sum_{\nu=0}^{n} d_{n\nu}a_\nu \mathfrak{P}_\nu(z)\right| \leqq \left|f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu}a_\nu/t^\nu\right|$$

$$+ \left|\sum_{\nu=1}^{n} (1 - d_{n\nu})a_\nu t \mathfrak{P}_\nu(t)\right|$$

$$+ \left|\sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t)\right|, \qquad r \leqq |t| \leqq r'.$$

The sum $\sum_{\nu=0}^{n} d_{n\nu}a_\nu/t^\nu$ is (SI) the Jackson summation of the first $n+1$ terms of the Taylor development of the function $f(\psi(t)) - \sum_1^\infty c_\nu t^\nu$ and consequently (Jackson, loc. cit.; Curtiss, loc. cit.)

$$\left|f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu}a_\nu/t^\nu\right| \leqq \frac{M_3}{n^{p+\alpha}}, \qquad r \leqq |t| \leqq r'.$$

Also

$$\left|\sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t)\right| \leqq \sum_{n+1}^{\infty} |a_\nu|\,|t|\,|\mathfrak{P}_\nu(t)| \leqq M_1 M_2 r \sum_{n+1}^{\infty}\left(\frac{r}{r'}\right)^\nu$$

$$\leqq \frac{M_4}{n^{p+\alpha}}, \qquad |t| = r,$$

since $r/r' < 1$. Furthermore*

$$|1 - d_{n\nu}| \leqq \frac{M_5 \nu^{p+1}}{n^{p+1}},$$

consequently

$$\left|\sum_{\nu=1}^{n} (1 - d_{n\nu})a_\nu t \mathfrak{P}_\nu(t)\right| \leqq M_1 M_2 M_5 \sum_{1}^{n} \frac{\nu^{p+1}}{n^{p+1}}\left(\frac{r}{r'}\right)^\nu$$

$$= \frac{M_1 M_2 M_5}{n^{p+1}} \sum_{1}^{n} \nu^{p+1}\left(\frac{r}{r'}\right)^\nu.$$

The series on the right converges, and the proof of the theorem is thus complete.

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* Dunham Jackson, Transactions of this Society, vol. 15 (1914), pp. 439–466. It should be noted here that Jackson's summation coefficients vary with the derivative but not with the order of the Lipschitz condition.