A NOTE ON THE ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS*

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Let $\psi(u)$ be a function non-decreasing in the interval (0, 1) such that all the moments

$$c_k = \int_0^1 u^k d\psi(u), \qquad k = 0, 1, 2, \cdots,$$

exist, and let c_0 be positive. Then

$$f(z) \equiv \int_0^1 \frac{d\psi(u)}{z-u} = \sum_{r=0}^{\infty} \frac{c_r}{z^{r+1}} \tag{1}$$

may be developed in a continued fraction of which the denominators of the successive approximants are the Tchebichef polynomials $Q_n(z)$,† where

$$\Delta_n Q_n(z) = \begin{vmatrix} c_0 & c_1 \cdots c_n \\ c_1 & c_2 \cdots c_{n+1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ c_{n-1} & c_n \cdots c_{2n-1} \\ 1 & z \cdots z^n \end{vmatrix}, \qquad \Delta_n = \begin{vmatrix} c_0 & c_1 \cdots c_n \\ c_1 & c_2 \cdots c_{n+1} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ c_{n-1} & c_n \cdots c_{2n-2} \end{vmatrix},$$

$$n = 0, 1, 2, \cdots; \Delta_0 = 1.$$

The determinants Δ_n are positive unless $\psi(u)$ has only a finite number ν of points of increase, in which case $\Delta_n = 0$ for $n > \nu$, and the continued fraction is terminating.

Shohat‡ has shown that for an extensive class of moment functions (1) we have

$$(\Delta_n/\Delta_{n+1})^{1/2} \sim 4^n, \tag{2}$$

and that for all functions of this type satisfying (2) the recurrence

* Presented to the Society, February 29, 1936.

† J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Paris, 1934, p. 12.

‡ J. Chokhatte (Shohat), *Sur le développement de l'intégrale $\int_a^b [p(y)/(x-y)]dy$ en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), p. 32.

relation connecting $Q_{n+2}(z)$, $Q_{n+1}(z)$, $Q_n(z)$ leads, as $n\to\infty$, to the characteristic equation

$$\lambda^2 - \lambda\left(z - \frac{1}{z}\right) + \frac{1}{16} = 0.* \tag{3}$$

From Poincaré's theorem† it follows that

$$\lim_{n\to\infty} \frac{Q_{n+1}(z)}{Q_n(z)}$$

exists and is equal to the root of larger modulus of (3) at all points z outside the range (0, 1) for which the moduli of the roots of (3) are equal. Shohat‡ has further shown that over the range (0, 1)

$$\lim_{n\to\infty} (\Delta_n/\Delta_{n+1})^{1/2} Q_n(z)/(1+\epsilon)^n = 0 \tag{4}$$

for every positive ϵ.

We prove the following theorem, which is a refinement of equation (4):

THEOREM 1. *For the type of function considered*

$$\lim_{n\to\infty} \max_{0\leqq z\leqq 1} \left| Q_n(z) \right|^{1/n} = \tfrac{1}{4}.$$

Following a theorem due to Perron,§ if the limiting form of the difference equation has roots of equal moduli, then although

$$\lim_{n\to\infty} \left\{ Q_{n+1}(z)/Q_n(z) \right\}$$

does not exist (in general),

$$\limsup_{n\to\infty} \left| Q_n(z) \right|^{1/n}$$

exists and is equal to the common absolute value.

At every point of (0, 1) both roots of (3) have the modulus 1/4. Hence, at every point of (0, 1)

$$\limsup_{n\to\infty} \left| Q_n(z) \right|^{1/n} = \tfrac{1}{4}. \tag{5}$$

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* Ibid., p. 43; see also Shohat, *Théorie Générale* · · · , pp. 50–52.

† N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924, pp. 300–305.

‡ See Shohat, *Sur le développement* · · · , p. 44; also *Théorie Générale* · · · , p. 52.

§ See Nörlund, p. 309.

Let $T_n(z)$ be the unique polynomial of degree n, with coefficient of z^n unity, whose maximum modulus over a given set E is a minimum. Fekete* has proved that

$$\lim_{n\to\infty} \max_{z\in E} \left| T_n(z) \right|^{1/n}$$

exists and is equal to the transfinite diameter of E.

It is well known that the transfinite diameter of the set (0, 1) is 1/4. Hence, in this case,

$$\lim_{n\to\infty} \max_{0\leqq z\leqq 1} \left| T_n(z) \right|^{1/n} = \tfrac{1}{4}. \tag{6}$$

From the uniqueness of $T_n(z)$ and (5) and (6), the result follows. It is of interest to note that the $Q_n(z)$, derived from $f(z)$, possess on the one hand the orthogonal properties of Tchebichef polynomials and on the other hand the properties of T-polynomials over the set of singularities of $f(z)$.

Formula (2) easily leads to the following result,† obtained by Pólya‡ in the case of a more restricted type of moment function:

THEOREM 2. *For the type of function considered*

$$\lim_{n\to\infty} \Delta^{1/n(n-1)} = \tfrac{1}{4}.$$

Theorems 1 and 2 are connected by the fact that each gives an expression§ for the transfinite diameter of the set of singularities of the moment function in terms of the coefficients of (1). The fundamental quantities $Q_n(z)$ and Δ_n there occurring are invariant under the substitutions $z' = z^{i\alpha} + c$, where α is real and c is, in general, complex.

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* M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahlingen Koeffizienten*, Mathematische Zeitschrift, vol. 17 (1923), pp. 234–236.

† See J. Shohat, *Théorie Générale* · · · , p. 57 for an associated theorem.

‡ G. Pólya, *Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe*, Mathematische Annalen, vol. 99 (1928), pp. 697–700.

§ See M. Fekete, loc. cit.