SOME INVARIANTS UNDER MONOTONE TRANSFORMATIONS*

D. W. HALL† AND A. D. WALLACE

We assume that $S$ is a locally connected, connected, compact metric space and that $P$ is a property of point sets. For any two points $a$ and $b$ of $S$ we denote by $C(ab)$ (respectively $C_i(ab)$) a closed (closed irreducible) cutting of $S$ between the points $a$ and $b$. We consider the following properties:

- $\Delta_0(P)$. If $S$ is the sum of two continua, their product has property $P$.
- $\Delta_1(P)$. If $K$ is a subcontinuum of $S$ and $R$ is a component of $S - K$, then the boundary of $R$, $(F(R) = \overline{R} - R)$, has property $P$.
- $\Delta_2(P)$. Each $C_i(ab)$ has property $P$.
- $\Delta_3(P)$. If $A$ and $B$ are disjoint closed sets containing the points $a$ and $b$, respectively, there is a $C(ab)$ disjoint from $A + B$ and having property $P$.

If $P$ is the property of being connected, the four properties $\Delta_i(P)$ are equivalent as shown by Kuratowski. Indeed it may be seen that Kuratowski’s proofs allow us to state the following theorem:

**Theorem 1.** For any property $P$ of point sets, $\Delta_i(P)$ implies $\Delta_{i+1}(P)$ for $i = 0, 1, 2$.

This result is the best possible in the sense that there is a property (that of being totally disconnected) for which no other implication holds.

The single-valued continuous transformation $T(S) = S'$ is said to be monotone if the inverse of every point is connected. It may be seen that the following statements are true:

(i) The inverse of every connected set is connected.
(ii) If the set $X$ separates $S$ between the inverses of the points $x$ and $y$, then $T(X)$ separates $S'$ between $x$ and $y.$

**Theorem 2. If the property $P$ is invariant under monotone trans-**
transformations, then for each \( i = 0, 1, 2, 3 \), the property \( \Delta_i(P) \) is invariant under the monotone transformation \( T(S) = S' \).

**Proof.** (0) If \( S' = L + M \), the summands being continua, then \( S = L^{-1} + M^{-1} \) is a sum of continua. Hence the set \( L^{-1} \cdot M^{-1} \) has property \( P \) and \( L \cdot M = T(L^{-1} \cdot M^{-1}) \) then has property \( P \).

(1) If \( R \) is a component of \( S' - K \), where \( K \) is a continuum, then \( R^{-1} \) is a component of the complement of the continuum \( K^{-1} \). By assumption, \( F(R^{-1}) \) has property \( P \). It follows that its image has property \( P \). But we have \( T(F(R^{-1})) = T(R^{-1} - R^{-1}) = T(R^{-1}) - R = F(R) \).

(2) Assume that \( C \) is a \( C(a, b) \) in \( S' \). From the continuity of \( T \) it follows that \( C^{-1} \) is a \( C(p, q) \) in \( S \), where \( a \) and \( b \) are any two points in the inverses of \( a \) and \( b \), respectively. Since the inverses of \( a \) and \( b \) are connected, there exists a cutting \( K \) of \( S \) between these two sets such that \( K \) is a \( C(x, y) \), where \( T(x) = a \) and \( T(y) = b \); and further \( K \) is a subset of \( C^{-1} \). Thus \( K \) has property \( P \); hence \( T(K) \) has. But \( T(K) \subseteq C \), and \( T(K) \) is a \( C(a, b) \). It follows that \( T(K) = C \) and from this that \( C \) has property \( P \).

(3) Let \( A \) and \( B \) denote disjoint closed subsets of \( S' \) containing \( a \) and \( b \). If \( x \) and \( y \) are points which map into \( a \) and \( b \), then by hypothesis there is a cutting \( K \) of \( S \) between \( x \) and \( y \) that is disjoint with \( A^{-1} \) and \( B^{-1} \) and has property \( P \). Since, clearly, \( K \) is a cutting of \( S \) between the inverses of \( a \) and \( b \), it follows that \( T(K) \) cuts \( S' \) between \( a \) and \( b \), is disjoint with \( A + B \), and has property \( P \).

As an application we have the following known results:

**Theorem 3.** The property of a locally connected continuum to be a dendrite, a regular curve, or a rational curve is a monotone invariant.

To see this we take \( P \) to be the property of being a point, a finite set of points, or a countable set of points and apply the invariance of \( \Delta_3(P) \).

**The University of Virginia**

* If \( X \) is a subset of \( S' \), we denote by \( X^{-1} \) the inverse of \( X \).
‡ See the fourth footnote and references given there.