ON THE DETERMINANT OF AN AUTOMORPH OF A NONSINGULAR SKEW-SYMMETRIC MATRIX

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Let \( G \) be the skew-symmetric matrix of order \( 2n \),

\[
G = \begin{pmatrix}
0 & E_n \\
- E_n & 0
\end{pmatrix},
\]

where \( E_n \) is the unit matrix of order \( n \). If \( F \) is a matrix which satisfies

\[
FGF' = G,
\]

then \( |F|^2 = 1 \), so that \( |F| = \pm 1 \). That \( |F| = +1 \) is well known and is in fact a consequence of a theorem of Frobenius.* A simple proof communicated to me by Professor Wintner depends on the polar factorization of \( F \), which reduces the problem at once to the case in which \( F \) is orthogonal. This proof is, of course, not valid in any field. It is our intention here to give a simple direct proof, applicable in any field, of the fact that \( |F| = +1 \).

On writing \( F \) as a matrix of matrices of orders, \( F = (F_{ij}) \), \((i, j = 1, 2)\), we have, as a consequence of (1),

\[
F_{11}F_{12} - F_{12}F_{11} = F_{21}F_{22} - F_{22}F_{21} = 0,
\]

\[
F_{11}F_{22} - F_{12}F_{21} = F_{22}F_{11} = F_{21}F_{12} = E.
\]

Let \( |F_{11}| \neq 0 \). Then

\[
F = \begin{pmatrix}
F_{11} & F_{12}F_{11}' \\
F_{21} & F_{22}F_{11}'
\end{pmatrix}
\begin{pmatrix}
E_n & 0 \\
0 & (F_{11}')^{-1}
\end{pmatrix}.
\]

On, applying (2), we have

\[
|F_{11}'| |F| = \begin{vmatrix}
F_{11} & F_{12}F_{11}' \\
F_{21} & F_{22}F_{11}'
\end{vmatrix}
\begin{vmatrix}
F_{11} & F_{12}F_{11}' - F_{11}F_{12}' \\
F_{21} & F_{22}F_{11}' - F_{21}F_{12}'
\end{vmatrix}
\begin{vmatrix}
F_{11} & 0 \\
F_{21} & E_n
\end{vmatrix}
= |F_{11}|.
\]

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Therefore \( |F| = +1 \), and we have proved the following lemma:

**Lemma 1.** If \( F_{11} \) is nonsingular, \( |F| = +1 \).

It also follows from (2) that if \( F_{12} = F_{21} = 0 \), \( F_{11}F_{22} = E_n \), so that \( F_{11} \) is nonsingular and accordingly \( |F| = +1 \).

Let \( P_{ij} \) be the permutation matrix of order \( n \), which by post-multiplication interchanges the \( i \)th and the \( j \)th columns of a matrix, and let \( P \) be the diagonal block matrix

\[
P = [P_{ij}, P_{ij}] = \begin{pmatrix} P_{ij} & 0 \\ 0 & P_{ij} \end{pmatrix}.
\]

Since \( P_{ij} \) is symmetric and involutory, \( PGP' = G \), the matrix \( FP \) satisfies (1), and \( |FP| = |F| \). Consequently we have the following lemma:

**Lemma 2.** Any matrix \( F_1 \) obtained from \( F \) by a permutation of its first \( n \) columns and the same permutation of its last \( n \) columns also satisfies (1) and \( |F_1| = |F| \).

The matrix \( W = (W_{ij}), (i, j = 1, 2) \), where \( W_{11} = W_{22} = [0, E_{n-1}] \) and \( W_{12} = -W_{21} = [E_1, 0] \), satisfies (1) and has determinant unity. The matrix \( FW \) is obtained from \( F \) by replacing the first column by minus the \((n+1)\)st column and the \((n+1)\)st by the first. If, for convenience, we now write \( F_{11} = A \) and \( F_{12} = B \) and denote the columns of \( A \) and \( B \) by \( a_i \) and \( b_i \), respectively, we have, as a consequence of Lemma 2, the following lemma:

**Lemma 3.** The matrix \( A = F_{11} \) in \( F \) may be replaced by \( C = (c_1, c_2, \ldots, c_n) \), where \( c_i = a_i \) or \( -b_i \).

Therefore by Lemma 1, since \( |W| = +1 \), we have our fourth lemma:

**Lemma 4.** If there exists a matrix \( C \) such that \( |C| \neq 0 \), then \( |F| = +1 \).

Let every determinant of order \( n \) formed from \((A, B)\), in which less than \( r \) pairs of columns \( a_i, b_i \) occur with the same suffix \( i \), be zero, but let at least one determinant with exactly \( r \) pairs of columns \( a_i, b_i \) be different from zero. As a consequence of Lemma 2 there is no loss in generality in assuming that

\[
(3) \quad |a_ib_iX| \neq 0,
\]

where the matrix \( X \) contains exactly \( r - 1 \) pairs of columns \( a_i, b_i \) with the same subscript \( i \) and does not contain either of the columns \( a_2 \) or \( b_2 \). Let \( Q \) be the diagonal block matrix
Then \( |Q| = +1 \), \( QGQ' = G \), and \( FQ \) satisfies (1). The matrix of the first \( n \) rows of \( FQ \) is \((H, K)\), where the \( n \) columns \( h_i \) of \( H \) are given by
\[
h_i = a_j, \quad j \neq 2, \quad h_2 = a_1 + a_2,
\]
and the \( n \) columns \( k_i \) of \( K \) by
\[
k_i = b_i, \quad i \neq 1, \quad k_1 = b_1 - b_2.
\]
Since the matrix \( X \) in (3) does not contain any of the columns \( a_3, b_1, a_2, b_2 \), it follows from (4) and (5) that the matrix \( T = (h_2 k_1 X) \) is a submatrix of \((H, K)\), which contains exactly \( r-1 \) pairs of columns \( h_i, k_i \) with the same suffix \( i \), and that
\[
|T| = |a_1 + a_2, b_1 - b_2, X| = |a_1 b_1 X| - |a_1 b_2 X| + |a_2 b_1 X| - |a_2 b_2 X|.
\]
But, by hypothesis,
\[
|a_1 b_2 X| = |a_2 b_1 X| = 0.
\]
Since \( |a_1 a_2 X| \) is also zero and \( |a_1 b_1 X| \) is not zero by (3), \( a_2 \) and \( b_2 \) are both linear combinations of the \( n-1 \) columns of the matrix \((a_1 X)\). Hence \(|a_2 b_2 X| = 0\) and, as a consequence of (6) and (7), \( |T| = |a_1 b_1 X| \neq 0\).

Therefore in \((H, K)\) there is one nonzero subdeterminant of order \( n \) which contains exactly \( r-1 \) pairs of columns \( h_i, k_i \) with the same suffix \( i \).

Now \( |FQ| = |F| \), and \( FQ \) satisfies (1). Further, the matrix \( C \) in Lemma 4 contains exactly \( r = 0 \) pairs of columns \( a_i, b_i \). By a simple induction proof we therefore have the following lemma:

**Lemma 5.** If in \((A, B)\) there is one nonzero subdeterminant of order \( n \) which contains \( r \leq n \) pairs of columns \( a_i, b_i \) with the same suffix \( i \), then \( |F| = +1 \).

Since any matrix \( F \) which satisfies (1) is nonsingular, the rank of \((A, B)\) is \( n \) and Lemma 5 implies the following statement:

**Theorem.** If \( FGF' = G \), \( |F| = +1 \).

This proof is valid in any field.