ON GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION IN SPHERICAL COORDINATES†

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In a previous paper,‡ the writer derived the expressions for the Green's functions in the theory of heat conduction for an infinite cylinder and for an infinite solid, bounded internally by a cylinder.

The object of the present paper is to derive the appropriate Green's functions for a sphere and for an infinite solid bounded internally by a sphere. In both cases, we shall take the boundary condition in the form

$$\frac{\partial u}{\partial r} + hu = 0, \qquad r = a.$$

The case of a sphere. In this case we start with the expression

$$u(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{2(\pi kt)^{3/2}} e^{-R^2/4kt}, \tag{1}$$

where

$$R^2 = r^2 + r_0^2 - 2r_0 \cos \gamma, \tag{2}$$

γ being the angle between the radii from the origin to the points (r, θ, ϕ) and (r_0, θ_0, ϕ_0). The expression (1) is the point source solution of the differential equation of heat conduction in spherical coordinates.

The expression (1) may be written in the form§

$$\begin{aligned} u(r, \theta, \phi, t; r_0, \theta_0; \phi_0) = \frac{1}{4\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) \\ \cdot \int_0^\infty \alpha e^{-k\alpha^2 t} J_{n+1/2}(\alpha r_0) J_{n+1/2}(\alpha r) d\alpha. \end{aligned} \tag{3}$$

The corresponding Laplace transform

$$L\{u(t)\} = \int_0^\infty e^{-pt} u(t) dt = u^*(p)$$

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‡ This Bulletin, vol. 44 (1938), pp. 125–133. This paper will be referred to as A.N.L.

§ See Carslaw, *Mathematical Theory of Heat Conduction*, article 93.

is therefore

$$u^*(r, \theta, \phi, p; r_0, \theta_0, \phi_0) = \frac{1}{4\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\gamma) \cdot \int_0^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_{n+1/2}(\alpha r)J_{n+1/2}(\alpha r_0), \tag{4}$$

where we have put $p = -kq^2$.

With the aid of the identities (5) and (5′) of A.N.L., (4) becomes

$$u^* = \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\gamma)J_{n+1/2}(rq)H^1_{n+1/2}(r_0q), \qquad r < r_0, \tag{5}$$

$$u^* = \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\gamma)J_{n+1/2}(r_0q)H^1_{n+1/2}(rq), \qquad r > r_0. \tag{6}$$

In order to obtain the Green's function, we must add to the point source solution u a function v, satisfying the differential equation of heat conduction, vanishing at $t=0$, and such that $u+v$ satisfies the boundary condition $\partial u/\partial r + hu = 0$, for $r=a$.

Since $L\{\partial u(t)/\partial t\} = L\{u(t)\} - u(0) = u^*(p) - u(0)$, and since the two operations of differentiation with respect to x, and of acting with the Laplace operator L, may be commuted, the Laplace transform of v must satisfy the differential equation

$$\Delta v^* + q^2 v^* = 0. \tag{7}$$

The transition from $u^* + v^*$ to the desired Green's function $G = u + v$. will be apparent from the subsequent developments.

The most general solution of (7) which is symmetric about the axis $\gamma = 0$ may be written in the form

$$v^* = \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1)A_nP_n(\cos\gamma)J_{n+1/2}(rq). \tag{8}$$

From (8) we get

$$\left(\frac{\partial v^*}{\partial r} + hv^*\right)_{r=a} = \frac{i}{8k(ar_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1)A_nP_n(\cos\gamma) \cdot \left\{ q\frac{d}{dz}J_{n+1/2}(z) + \left[h - \frac{1}{2a}\right]J_{n+1/2}(z)\right\}_{z=aq}. \tag{9}$$

Since

$$\left(\frac{\partial}{\partial r} + h\right)(u^* + v^*) = 0, \qquad r = a, \tag{10}$$

it follows that

$$(11)\quad A_n = -J_{n+1/2}(r_0 q)\frac{\left\{q\dfrac{d}{dz}H^1_{n+1/2}(z) + (h - 1/(2a))H^1_{n+1/2}(z)\right\}_{z=aq}}{qJ'_{n+1/2}(aq) + (h - 1/(2a))J_{n+1/2}(aq)};$$

therefore†

$$(12)\qquad u^* + v^* = \frac{i}{8k(rr_0)^{1/2}}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma)w_n^*,$$

where

$$(13)\qquad \begin{aligned} w_n^* = \frac{J_{n+1/2}(rq)}{U_{n+1/2}(aq)}\Bigg\{&H^1_{n+1/2}(rq)\,U_{n+1/2}(aq) - J_{n+1/2}(rq) \\ &\cdot\left[\frac{z}{a}\,\frac{d}{dz}H^1_{n+1/2}(z) + \left(h - \frac{1}{2a}\right)H^1_{n+1/2}(z)\right]_{z=aq}\Bigg\} \end{aligned}$$

and

$$(14)\qquad U_{n+1/2}(aq) = qJ'_{n+1/2}(aq) + \left(h - \frac{1}{2a}\right)J_{n+1/2}(aq).$$

Comparison between (14) and equation (14) of A.N.L. shows clearly that there is a formal analogy between the present and the former expression for w_n^*. Specifically, our present w_n^* may be obtained from the corresponding expression in A.N.L. by replacing n by $n+1/2$ and h by $h-1/(2a)$ and multiplying by the factor $1/2$. The inversion of (12) therefore ultimately yields‡

$$(15)\qquad \begin{aligned} G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) &= \frac{1}{2\pi a^2(r\,r_0)^{1/2}}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\gamma) \\ &\cdot\sum_{q_i} q_i^2 e^{-kq_i^2 t}J_{n+1/2}(q_i r) \\ &\cdot\frac{J_{n+1/2}(q_i r_0)}{[(h - 1/(2a))^2 + q_i^2 - (n+1/2)^2/a^2][J_{n+1/2}(q_i a)]^2}, \end{aligned}$$

† Formulas (13), (15), (18), (19), (20), and (22) are given for $r<r_0$. In the case $r>r_0$, the corresponding formulas are obtained by interchanging r and r_0.

‡ As mentioned in A.N.L., the transition from $pw_n^* = Y(p)/Z(p)$ to w_n is equivalent to the inversion of the Laplace transform defining w_n^*, and we have

$$w_n = \frac{Y(0)}{Z(0)} + \sum\frac{Y(p_i)}{p_i Z'(p_i)}\cdot e^{p_i t},$$

where the summation extends over the roots of $Z(p)=0$.

where the second summation extends over the roots

$$U_{n+1/2}(aq) = 0. \tag{16}$$

From this formula we may obtain the Green's function for the case where the boundary is impervious to heat by putting $h=0$. Also the case where the boundary is kept at 0° may be obtained by putting $h=\infty$. In this case it is clear that the transcendental equation (16) reduces to

$$J_{n+1/2}(aq) = 0. \tag{17}$$

Also it is easily seen that the denominator of (15) reduces to

$$q_i^2 \, [J'_{n+1/2}(q_i a)]^2.$$

Thus the Green's function for the case where the boundary is kept at 0° is

$$\begin{aligned} G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{2\pi a^2 (r\, r_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) \sum_{q_i} e^{-kq_i^2 t} \\ \cdot \frac{J_{n+1/2}(q_i r) J_{n+1/2}(q_i r_0)}{\{J'_{n+1/2}(aq_i)\}^2}, \end{aligned} \tag{18}$$

where the second summation extends over the roots of (17).

Case of the infinite solid bounded internally by a sphere. The former analogy with the treatment in A.N.L., noticed in the previous case, applies also in the case under consideration. Thus since v^* must be finite for $r=\infty$, it follows that in (8) we must replace $J_{n+1/2}(rq)$ by $H^1_{n+1/2}(rq)$. Proceeding as in the previous case, we ultimately obtain

$$u^* + v^* = \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) W_n^*, \tag{19}$$

where the expression for W_n^* may be obtained from equation (30) of A.N.L. by replacing h by $h-1/(2a)$ and n by $n+1/2$ and multiplying by the factor 1/2. Our final solution is therefore

$$\begin{aligned} G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\gamma) \\ \cdot \int_{-\infty}^{+\infty} \alpha e^{-k\alpha^2 t} \frac{H^1_{n+1/2}(\alpha r_0)}{U_{n+1/2}(\alpha a)} \\ \cdot \{J_{n+1/2}(\alpha r)\, U_{n+1/2}(\alpha a) - U_{n+1/2}(\alpha r) J_{n+1/2}(\alpha a)\} d_\alpha \end{aligned} \tag{20}$$

where

$$U_{n+1/2}(\alpha a) = \left\{\alpha \frac{d}{dz} H^1_{n+1/2}(z) + \left(h - \frac{1}{2a}\right) H^1_{n+1/2}(z)\right\}_{z=a\alpha}. \tag{21}$$

For $h=\infty$ this reduces to

$$\begin{aligned} G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \\ \cdot \int_{-\infty}^{+\infty} \alpha e^{-k\alpha^2 t} \frac{H^1_{n+1/2}(\alpha r_0)}{H^1_{n+1/2}(\alpha a)} \\ \cdot \left\{J_n(\alpha r) H^1_{n+1/2}(\alpha a) - J_{n+1/2}(\alpha a) H^1_{n+1/2}(\alpha r)\right\} d\alpha. \end{aligned} \tag{22}$$

This is the solution of our problem when the spherical surface $r=a$ is kept at 0°.

The Green's functions above evaluated may be called *point source Green's functions*. They are solutions of the differential equation of heat conduction, depending on the spherical coordinates r, θ, and ϕ and satisfying the condition

$$\lim_{\epsilon\to 0} \iiint_{\omega} G(r, \theta, \phi, 0; r', \theta', \phi') d\tau = 1, \tag{23}$$

where ω is a little sphere of radius ϵ surrounding the point source (r_0, θ_0, ϕ_0).

In addition to these Green's functions we may consider the Green's functions depending on r only and satisfying the condition

$$\lim_{\epsilon\to 0} 4\pi \int_{r_0}^{r_0+\epsilon} G(r, \rho, 0) \rho^2 d\rho = 1. \tag{24}$$

For the case of the sphere radiating into a medium at 0°, the Green's function, while not given explicitly by Carslaw, may be derived from his article 65, in the form

$$\begin{aligned} G(r, t; r_0) = \frac{1}{2\pi a r r_0} \sum_{n=1}^{\infty} \frac{a^2\alpha_n^2 + (ah - 1)^2}{a^2\alpha_n^2 + ah(ah - 1)} \\ \cdot \sin \alpha_n r \cdot \sin \alpha_n r_0 e^{-k\alpha_n^2 t}, \end{aligned} \tag{25}$$

where α_n is a root of $a\alpha \cos a\alpha + (ah-1) \sin a\alpha = 0$.

The Green's function for the case of the infinite solid bounded internally by a sphere may be obtained by considering a continuous distribution of point sources over the sphere $r=r_0$ and integrating for the variables θ' and ϕ'. This leads to

$$G(r, t; r_0) = \frac{1}{2\pi a^2} \sum_{q_i} P_0(\cos \gamma) e^{-kq_i^2 t} \cdot \frac{J_{1/2}(q_i r) J_{1/2}(q_i r_0)}{[(h - 1/(2a))^2 + q_i^2 - 1/(4a^2)][J_{1/2}(q_i a)]^2}, \tag{26}$$

where the summation extends over the roots of (17).

The desired results may also be obtained in the following manner. It can be easily shown that if $u(r, \rho, t)$, is the Green's function appropriate to a "plane source," and therefore satisfying the condition

$$\lim_{\epsilon \to 0} \int_{r_0}^{r_0+\epsilon} u(r, \rho, 0) d\rho = 1, \tag{27}$$

then

$$v = \frac{1}{4\pi r r_0} u$$

is the desired Green's function appropriate to a spherical source. By substituting for u the expression which may be derived from Carslaw's article 82, the desired Green's function is obtained in the form

$$\begin{aligned} G(r, t; r_0) = \frac{1}{8\pi r r_0 (\pi k t)^{1/2}} \Bigg\{ &\exp\left[- \frac{(r - a - r_0)^2}{4kt} \right] \\ &+ \exp\left[- \frac{(r - a + r_0)^2}{4kt} \right] \\ &- 2h \int_0^\infty e^{-h\xi} \exp\left[- \frac{(r - a + r_0 + \xi)^2}{4kt} \right] d\xi \Bigg\} \end{aligned} \tag{28}$$

which must, of course, agree with (26).

It should be remarked that the Green's functions so derived are of the general form

$$G = \sum u_n(P) \cdot u_n(P_0) e^{-k\lambda_n^2 t}, \tag{29}$$

where the u_n's are the normalized characteristic solutions of the homogeneous differential equation of

$$\nabla^2 u + \lambda^2 u = 0 \tag{30}$$

which satisfies the prescribed boundary conditions.

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