ON GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION IN SPHERICAL COORDINATES†

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In a previous paper,‡ the writer derived the expressions for the Green's functions in the theory of heat conduction for an infinite cylinder and for an infinite solid, bounded internally by a cylinder.

The object of the present paper is to derive the appropriate Green's functions for a sphere and for an infinite solid bounded internally by a sphere. In both cases, we shall take the boundary condition in the form

\[ \frac{\partial u}{\partial r} + hu = 0, \quad r = a. \]

The case of a sphere. In this case we start with the expression

\[ u(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{2(\pi kt)^{3/2}} e^{-R^2/4kt}, \]

where

\[ R^2 = r^2 + r_0^2 - 2r_0 \cos \gamma, \]

\( \gamma \) being the angle between the radii from the origin to the points \((r, \theta, \phi)\) and \((r_0, \theta_0, \phi_0)\). The expression (1) is the point source solution of the differential equation of heat conduction in spherical coordinates.

The expression (1) may be written in the form§

\[ u(r, \theta, \phi, t; r_0, \theta_0; \phi_0) = \frac{1}{4\pi (rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \cdot \int_0^\infty e^{-ka^2} J_{n+1/2}(ar_0) J_{n+1/2}(ar) d\alpha. \]

The corresponding Laplace transform

\[ L\{u(t)\} = \int_0^\infty e^{-p\tau} u(t) dt = u^*(p) \]

† Presented to the Society, October 29, 1938.
‡ This Bulletin, vol. 44 (1938), pp. 125–133. This paper will be referred to as A.N.L.
§ See Carslaw, Mathematical Theory of Heat Conduction, article 93.
is therefore
\[
\begin{align*}
\nu^*(r, \theta, \phi, p; r_0, \theta_0, \phi_0) &= \frac{1}{4\pi(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \gamma) \\
&\quad \cdot \int_0^\infty \frac{\alpha d\alpha}{\alpha^2 - q^2} J_{n+1/2}(\alpha r)J_{n+1/2}(\alpha r_0),
\end{align*}
\]
where we have put \( p = -kq^2 \).

With the aid of the identities (5) and (5') of A.N.L., (4) becomes
\[
\begin{align*}
(5) \quad \nu^* &= \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \gamma)J_{n+1/2}(rq)H_{n+1/2}(r_0q), \quad r < r_0, \\
(6) \quad \nu^* &= \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1)P_n(\cos \gamma)J_{n+1/2}(r_0q)H_{n+1/2}(rq), \quad r > r_0.
\end{align*}
\]

In order to obtain the Green’s function, we must add to the point source solution \( \nu \) a function \( v \), satisfying the differential equation of heat conduction, vanishing at \( t = 0 \), and such that \( \nu + v \) satisfies the boundary condition \( \partial \nu / \partial r + hv = 0 \), for \( r = a \).

Since \( L \{ \partial \nu(t) / \partial t \} = L \{ \nu(t) \} - \nu(0) = \nu^*(p) - \nu(0) \), and since the two operations of differentiation with respect to \( x \), and of acting with the Laplace operator \( L \), may be commuted, the Laplace transform of \( v \) must satisfy the differential equation
\[
\Delta v^* + q^2 v^* = 0.
\]

The transition from \( \nu^* + v^* \) to the desired Green’s function \( G = \nu + v \), will be apparent from the subsequent developments.

The most general solution of (7) which is symmetric about the axis \( \gamma = 0 \) may be written in the form
\[
\nu^* = \frac{i}{8k(rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1)A_nP_n(\cos \gamma)J_{n+1/2}(rq).
\]

From (8) we get
\[
\begin{align*}
\left( \frac{\partial \nu^*}{\partial r} + hv^* \right)_{r=a} &= \frac{i}{8k(ar_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1)A_nP_n(\cos \gamma) \\
&\quad \cdot \left\{ q \frac{d}{dz} J_{n+1/2}(z) + \left[ h - \frac{1}{2a} \right] J_{n+1/2}(z) \right\}_{z=aq}.
\end{align*}
\]

Since
\[
\left( \frac{\partial}{\partial r} + h \right)(\nu^* + v^*) = 0, \quad r = a,
\]
it follows that
\[ A_n = - J_{n+1/2}(rq) \frac{qJ'_{n+1/2}(aq) + (h - 1/(2a))J_{n+1/2}(aq)}{qJ'_{n+1/2}(aq) + (h - 1/(2a))J_{n+1/2}(aq)} ; \]

therefore†
\[ u^* + v^* = \frac{i}{8k(r_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) w_n^* , \]
where
\[ w_n^* = \frac{J_{n+1/2}(rq)}{U_{n+1/2}(aq)} \left\{ H_{n+1/2}(rg) U_{n+1/2}(aq) - J_{n+1/2}(rg) \right\} \]
\[ \cdot \left[ \frac{z}{a} \frac{d}{dz} H_{n+1/2}(z) + \left( h - \frac{1}{2a} \right) H_{n+1/2}(z) \right]_{z=aq} \]
and
\[ U_{n+1/2}(aq) = qJ'_{n+1/2}(aq) + \left( h - \frac{1}{2a} \right) J_{n+1/2}(aq) . \]

Comparison between (14) and equation (14) of A.N.L. shows clearly that there is a formal analogy between the present and the former expression for \( w_n^* \). Specifically, our present \( w_n^* \) may be obtained from the corresponding expression in A.N.L. by replacing \( n \) by \( n + 1/2 \) and \( h \) by \( h - 1/(2a) \) and multiplying by the factor 1/2. The inversion of (12) therefore ultimately yields‡
\[ G(r, \theta, \phi; t; r_0, \theta_0, \phi_0) = \frac{1}{2\pi a^2 (r r_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \]
\[ \cdot \sum_{q_i} q_i^2 e^{-kq_i^2} J_{n+1/2}(q_i r) \frac{J_{n+1/2}(q_i r_0)}{\left[ (h - 1/(2a))^2 + q_i^2 - (n + 1/2)^2/a^2 \right]^{3/2}} , \]

† Formulas (13), (15), (18), (19), (20), and (22) are given for \( r < r_0 \). In the case \( r > r_0 \), the corresponding formulas are obtained by interchanging \( r \) and \( r_0 \).
‡ As mentioned in A.N.L., the transition from \( p w_n^* = Y(p)/Z(p) \) to \( w_n \) is equivalent to the inversion of the Laplace transform defining \( w_n^* \), and we have
\[ w_n = Y(0) / Z(0) + \sum_{p|Z'(p)} Y(p) e^{pvt} , \]
where the summation extends over the roots of \( Z(p) = 0 \).
where the second summation extends over the roots

\[ U_{n+1/2}(aq) = 0. \]  

From this formula we may obtain the Green's function for the case where the boundary is impervious to heat by putting \( h = 0 \). Also the case where the boundary is kept at \( 0^\circ \) may be obtained by putting \( h = \infty \). In this case it is clear that the transcendental equation (16) reduces to

\[ J_{n+1/2}(aq) = 0. \]  

Also it is easily seen that the denominator of (15) reduces to

\[ q^2 \left[ J'_{n+1/2}(aq) \right]^2. \]

Thus the Green's function for the case where the boundary is kept at \( 0^\circ \) is

\[ G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{2\pi a^2(r_0)_{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \sum_{\xi} e^{-kq^2t} \]

\[ \cdot \frac{J_{n+1/2}(aq) J_{n+1/2}(aq_0)}{\left\{ J'_{n+1/2}(aq) \right\}^2}, \]

where the second summation extends over the roots of (17).

**Case of the infinite solid bounded internally by a sphere.** The former analogy with the treatment in A.N.L., noticed in the previous case, applies also in the case under consideration. Thus since \( v^* \) must be finite for \( r = \infty \), it follows that in (8) we must replace \( J_{n+1/2}(rq) \) by \( H^1_{n+1/2}(rq) \). Proceeding as in the previous case, we ultimately obtain

\[ u^* + v^* = \frac{i}{8k(\pi)_{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) W_n^*, \]

where the expression for \( W_n^* \) may be obtained from equation (30) of A.N.L. by replacing \( h \) by \( h - 1/(2a) \) and \( n \) by \( n + 1/2 \) and multiplying by the factor \( 1/2 \). Our final solution is therefore

\[ G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi(\pi)_{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \]

\[ \cdot \int_{-\infty}^{+\infty} e^{-k\alpha^2t} \frac{H^1_{n+1/2}(\alpha r_0)}{U_{n+1/2}(\alpha a)} \]

\[ \cdot \left\{ J_{n+1/2}(\alpha r) U_{n+1/2}(\alpha a) - U_{n+1/2}(\alpha r) J_{n+1/2}(\alpha a) \right\} d\alpha \]
where

\[ U_{n+1/2}(\alpha a) = \left\{ \alpha \frac{d}{dz} H_{n+1/2}(z) + \left( h - \frac{1}{2a} \right) H_{n+1/2}(z) \right\}_{z=a} \]  

For \( h = \infty \) this reduces to

\[ G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi (rr_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma) \]

\[ \int_{-\infty}^{+\infty} \alpha e^{-k\alpha^2 t} \frac{H_{n+1/2}(\alpha r_0)}{H_{n+1/2}(\alpha a)} \left\{ J_n(\alpha r) H_{n+1/2}(\alpha a) - J_{n+1/2}(\alpha a) H_{n+1/2}(\alpha r) \right\} d\alpha. \]

This is the solution of our problem when the spherical surface \( r = a \) is kept at 0°.

The Green's functions above evaluated may be called \textit{point source Green's functions}. They are solutions of the differential equation of heat conduction, depending on the spherical coordinates \( r, \theta, \) and \( \phi \) and satisfying the condition

\[ \lim_{\epsilon \to 0} \int \int \int_{\omega} G(r, \theta, \phi, 0; r', \theta', \phi') dr = 1, \]

where \( \omega \) is a little sphere of radius \( \epsilon \) surrounding the point source \((r_0, \theta_0, \phi_0)\).

In addition to these Green's functions we may consider the Green's functions depending on \( r \) only and satisfying the condition

\[ \lim_{\epsilon \to 0} 4\pi \int_{r_0}^{r_0+\epsilon} G(r, \rho, 0)\rho^2 d\rho = 1. \]

For the case of the sphere radiating into a medium at 0°, the Green's function, while not given explicitly by Carslaw, may be derived from his article 65, in the form

\[ G(r, t; r_0) = \frac{1}{2\pi ar_0} \sum_{n=1}^{\infty} \frac{a^2 \alpha_n^2 + (ah - 1)^2}{a^2 \alpha_n^2 + ah(ah - 1)} \sin \alpha_n r \cdot \sin \alpha_n r_0 e^{-k\alpha_n^2 t}, \]

where \( \alpha_n \) is a root of \( a\alpha \cos a\alpha + (ah - 1)\sin a\alpha = 0. \)

The Green's function for the case of the infinite solid bounded internally by a sphere may be obtained by considering a continuous distribution of point sources over the sphere \( r = r_0 \) and integrating for the variables \( \theta' \) and \( \phi' \). This leads to
\[ G(r, t; r_0) = \frac{1}{2\pi a^2} \sum q_i P_0(\cos \gamma) e^{-k a^2 t} \]

\[ \frac{J_{1/2}(q r) J_{1/2}(q r_0)}{[(h - 1/(2a))^2 + q^2 - 1/(4a^2)] [J_{1/2}(q a)]^2}, \]

where the summation extends over the roots of (17).

The desired results may also be obtained in the following manner. It can be easily shown that if \( u(r, \rho, t) \), is the Green’s function appropriate to a “plane source,” and therefore satisfying the condition

\[ \lim_{\epsilon \to 0} \int_{r_0}^{r_0+\epsilon} u(r, \rho, 0) d\rho = 1, \]

then

\[ v = \frac{1}{4\pi r r_0} u \]

is the desired Green’s function appropriate to a spherical source. By substituting for \( u \) the expression which may be derived from Carslaw’s article 82, the desired Green’s function is obtained in the form

\[ G(r, t; r_0) = \frac{1}{8\pi r r_0 (\pi k t)^{1/2}} \left\{ \exp \left[ - \frac{(r - a - r_0)^2}{4kt} \right] \right. \]

\[ + \exp \left[ - \frac{(r - a + r_0)^2}{4kt} \right] \]

\[ - 2h \int_{0}^{\infty} e^{-k \xi} \exp \left[ - \frac{(r - a + r_0 + \xi)^2}{4kt} \right] d\xi \]

which must, of course, agree with (26).

It should be remarked that the Green’s functions so derived are of the general form

\[ G = \sum u_n(P) \cdot u_n(P_0) e^{-k a^2 t}, \]

where the \( u_n \)'s are the normalized characteristic solutions of the homogeneous differential equation of

\[ \nabla^2 u + \lambda^2 u = 0 \]

which satisfies the prescribed boundary conditions.

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