A TENSOR ANALYSIS FOR A V_k IN A PROJECTIVE SPACE S_n *

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1. **Introduction.** In this paper we shall show how an intrinsic tensor analysis may be developed for a curved space or variety V_k of k dimensions immersed in a projective space S_n of $n>k>1$ dimensions. For $n\neq k-1$ there apparently exists no covariant quadratic differential form; so Fubini's method of studying such a variety fails. Or, as Lane suggests,† Fubini's method fails either due to the lack of a quadratic differential form, or to the lack of an absolute calculus for an n-ary p-adic form except when $p=2$.

However, it is well known that an absolute calculus can be developed without the use of a quadratic form by making use of certain generalized Christoffel symbols.‡

These three indexed symbols enable one to introduce into the geometrical theory of a variety the geometry of paths, affine and "projective" connections. In that manner certain tensors and vectors arising in those theories can be expressed in terms of tensors and vectors arising in the study of the variety from the point of view of classical projective geometry. In particular, the Weyl projective curvature tensor is expressible in terms of tensors arising in the classical geometric theory of a variety V_k.

Finally, we show that a generalized Riemann space of k dimensions with a fundamental symmetric connection characterizing the space may be considered as being immersed in a projective space of $n=k(k+3)/2$ dimensions. This theorem is an evident generalization of the fact that a Riemann space may always be considered as immersed in an euclidean space of sufficiently high dimension.

2. **The fundamental differential equations.** Let the homogeneous projective coordinates x^i, $(i=1, 2, \cdots, n+1)$, of a point P in S_n be given as analytic functions of exactly k parameters $u^1, u^2, \cdots, u^k$:

$$x^i = x^i(u^1, u^2, \cdots, u^k). \tag{1}$$

The totality of such points P we shall call *a variety* V_k.

The functions x and $\partial x/\partial u^\rho$ may be interpreted as the homogeneous projective coordinates of $k+1$ points. These points determine a cer-

* An address delivered before the Cleveland meeting of the Society on November 25, 1938, by invitation of the Program Committee.

† Lane [**1**, p. 292].

‡ See, for example, Eisenhart [**2**].

tain linear space T_k of k dimensions which we shall call the *tangent space of V_k at x.*

Let ${}_{(s)}y^i$, $(s=1, 2, \cdots, r=n-k)$, be $r(n+1)$ other functions of $u^1, u^2, \cdots, u^k$ such that the determinant

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y\right) \tag{2}$$

does not vanish. Then the $n+1$ functions x and the $r(n+1)$ functions ${}_{(s)}y$ satisfy a system of differential equations of the form

$$\begin{aligned} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} &= L^\rho_{\alpha\beta} \frac{\partial x}{\partial u^\rho} + p_{\alpha\beta} x + {}_{(s)}D_{\alpha\beta}\, {}_{(s)}y, \\ \frac{\partial\, {}_{(s)}y}{\partial u^\alpha} &= {}_{(s)}M^\rho_\alpha \frac{\partial x}{\partial u^\rho} + {}_{(s)}q_\alpha x + {}_{(st)}E_\alpha\, {}_{(t)}y. \end{aligned} \tag{3}$$

We shall use the usual umbral convention: any index repeated in any symbol or group of symbols denotes summation over that index. The Greek letters $\alpha, \beta, \gamma, \cdots, \rho, \sigma$ shall be understood to have the range from 1 to k, the letters i, j from 1 to $n+1$, and the letters s, p, l, m from 1 to r.

From (3) we note that $L^\rho_{\alpha\beta} = L^\rho_{\beta\alpha}$, $p_{\alpha\beta} = p_{\beta\alpha}$, ${}_{(s)}D_{\alpha\beta} = {}_{(s)}D_{\beta\alpha}$. Moreover any other solution X^i, ${}_{(s)}Y^i$ of (3) is of the form

$$X^i = a^i_j x^j, \qquad {}_{(s)}Y^i = a^i_j\, {}_{(s)}y^j, \qquad a = |a^i_j| \neq 0. \tag{4}$$

Hence the coefficients of system (3) are invariant under the projective transformation (4).

The coefficients of system (3) are not independent. They satisfy certain integrability conditions. These conditions are

$$\begin{aligned} P^\rho_{\alpha\beta\gamma} &= R^\rho_{\alpha\beta\gamma} + p_{\alpha\gamma}\delta^\rho_\beta - p_{\alpha\beta}\delta^\rho_\gamma \\ &= {}_{(s)}D_{\alpha\beta}\, {}_{(s)}M^\rho_\gamma - {}_{(s)}D_{\alpha\gamma}\, {}_{(s)}M^\rho_\beta, \\ p_{\alpha\beta,\gamma} - p_{\alpha\gamma,\beta} &= {}_{(s)}q_\beta\, {}_{(s)}D_{\alpha\gamma} - {}_{(s)}q_\gamma\, {}_{(s)}D_{\alpha\beta}, \\ {}_{(s)}D_{\alpha\beta,\gamma} - {}_{(s)}D_{\alpha\gamma,\beta} &= {}_{(ts)}E_\beta\, {}_{(t)}D_{\alpha\gamma} - {}_{(ts)}E_\gamma\, {}_{(t)}D_{\alpha\beta}, \\ {}_{(s)}M^\rho_{\alpha,\beta} - {}_{(s)}M^\rho_{\beta,\alpha} &= \delta^\rho_\alpha\, {}_{(s)}q_\beta - \delta^\rho_\beta\, {}_{(s)}q_\alpha \\ &\quad + {}_{(st)}E_\beta\, {}_{(t)}M^\rho_\alpha - {}_{(st)}E_\alpha\, {}_{(t)}M^\rho_\beta, \\ {}_{(s)}q_{\alpha,\beta} - {}_{(s)}q_{\beta,\alpha} &= {}_{(st)}E_\beta\, {}_{(t)}q_\alpha - {}_{(st)}E_\alpha\, {}_{(t)}q_\beta \\ &\quad + p_{\alpha\sigma}\, {}_{(s)}M^\sigma_\beta - p_{\beta\sigma}\, {}_{(s)}M^\sigma_\alpha, \\ {}_{(st)}E_{\alpha,\beta} - {}_{(st)}E_{\beta,\alpha} &= {}_{(sp)}E_\beta\, {}_{(pt)}E_\alpha - {}_{(sp)}E_\alpha\, {}_{(pt)}E_\beta \\ &\quad + {}_{(t)}D_{\alpha\rho}\, {}_{(s)}M^\rho_\beta - {}_{(t)}D_{\rho\beta}\, {}_{(s)}M^\rho_\alpha, \end{aligned} \tag{5}$$

wherein

$$(6) \qquad R^{\rho}_{\alpha\beta\gamma} = \frac{\partial}{\partial u^{\beta}} L^{\rho}_{\alpha\gamma} - \frac{\partial}{\partial u^{\gamma}} L^{\rho}_{\alpha\beta} + L^{\sigma}_{\alpha\gamma} L^{\rho}_{\sigma\beta} - L^{\sigma}_{\alpha\beta} L^{\rho}_{\sigma\gamma}.$$

The other abbreviations occurring in (5) will be explained at a later time.

By direct calculation from the integrability conditions or by proper changes in notation in a paper by Lane,* we find that

$$(7) \qquad \frac{\partial}{\partial u^{\alpha}} [L^{\rho}_{\rho\beta} + {}_{(ss)}E_{\beta}] = \frac{\partial}{\partial u^{\beta}} [L^{\rho}_{\rho\alpha} + {}_{(ss)}E_{\alpha}].$$

Hence there exists a function f such that

$$(8) \qquad f_{\sigma} = \frac{\partial f}{\partial u^{\sigma}} = L^{\rho}_{\rho\sigma} + {}_{(ss)}E_{\sigma}.$$

3. **Tensors on** V_k. The variety V_k is not changed if on the differential equations (3) we make the transformations

$$(9) \qquad u^{\alpha} = u^{\alpha}(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^k), \qquad \Delta = \left| \frac{\partial u^{\alpha}}{\partial \bar{u}^{\beta}} \right| \neq 0;$$

$$(10) \qquad x = \lambda \bar{x}, \qquad \lambda \neq 0;$$

$$(11) \qquad {}_{(s)}y = {}_{(s)}\theta^{\rho} \frac{\partial x}{\partial u^{\rho}} + {}_{(s)}\phi x + a_{st}\, {}_{(t)}\bar{y}, \qquad A = | a_{st} | \neq 0.$$

Under (9) the differential equations (3) assume a form wherein the new coefficients $\bar{L}^{\rho}_{\alpha\beta}$ are given by

$$(12) \qquad \bar{L}^{\rho}_{\alpha\beta} = \left[L^{\lambda}_{\gamma\delta} \frac{\partial u^{\gamma}}{\partial \bar{u}^{\alpha}} \frac{\partial u^{\delta}}{\partial \bar{u}^{\beta}} + \frac{\partial^2 u^{\lambda}}{\partial \bar{u}^{\alpha} \partial \bar{u}^{\beta}} \right] \frac{\partial \bar{u}^{\rho}}{\partial u^{\lambda}}.$$

Moreover the coefficients $p_{\alpha\beta}$, ${}_{(s)}D_{\alpha\beta}$ transform by (9) like the covariant components of tensors of the second order; and ${}_{(s)}q_{\alpha}$, ${}_{(st)}E_{\alpha}$ transform like the covariant components of vectors (or tensors of the first order); and ${}_{(s)}M^{\rho}_{\alpha}$ transforms like the components of a mixed tensor of the second order. Accordingly, we call them the contravariant and covariant components of tensors of the kind and order indicated by their indices.

We may readily verify that $P^{\rho}_{\alpha\beta\gamma}$, $R^{\rho}_{\alpha\beta\gamma}$ are components of tensors of the fourth order, contravariant of the first, and covariant of the

* Lane [3, p. 796].

third. We shall call the tensor $P^{\rho}_{\alpha\beta\gamma}$ *the projective curvature tensor of* V_k *relative to the space* N_r of points x, ${}_{(s)}y$.

Unfortunately these vectors and tensors have no geometric significance since, although invariant under the projective transformation (4), they are not invariant under (10) and (11). To this end we normalize the coordinates x themselves, and the space of points N_r determined by x and ${}_{(s)}y$.

4. **A semi-canonical form of the differential equations.** Differential geometers have used various methods for reducing the differential equations, forms, or power series arising in their respective theories to canonical forms. In the case $k=2$, $r=1$ (that is, in the case of an ordinary surface in three dimensional projective space S_3), Fubini normalized the coordinates x by a special transformation (10) which made the ratio of the discriminants of certain covariant differential forms (one quadratic and one cubic) a constant.* In the case $k=n-1$. $r=1$ (that is, in the case of a hypersurface in S_n), Hlavatý used one covariant quadratic form, covariant differentiation with respect to the form, and the properties of affine connection to derive a canonical form of his differential equations.†

Wilczynski, starting with a defining set of differential equations, observed the effect of transformations of the arbitrary parameters in the differential equations which left invariant the configuration he was studying, and then by a judicious particular choice of transformation reduced his set to a canonical form.‡ He also always computed a complete system of invariants and covariants for his configuration. A variation of his method for $k=2$, $r=1$ consists in starting with a Taylor's expansion of one nonhomogeneous projective coordinate in terms of the other two, and then, by making use of available transformations of the arbitrary parameters, reducing the power series to a simple or convenient canonical form. A weather eye was always held out, however, for a form to which a more or less simple geometrical significance could be attached. This method§ has been used extensively by Lane, Stouffer, Green, and others.

We use an adaptation|| of the method used by Grove for reducing our defining differential equations to a canonical form. We observe first that the functions f_σ defined by (8) transform by (9), (10), (11) into $\bar{f}_{\bar{\sigma}}$ by the formula

* Fubini and Čech [**4, 5**].

† Hlavatý [**6**].

‡ Wilczynski [**7, 8, 9**].

§ Lane [**10**].

|| Grove [**11**].

$$\bar{f}_{\bar{\sigma}} = \frac{\partial u^\rho}{\partial \bar{u}^\sigma}\left[f_\rho + \frac{\partial}{\partial u^\rho}\log\frac{\Delta}{A\lambda^{k+1}}\right]. \tag{13}$$

Let R be a function of $u^1, u^2, \cdots, u^k$ with the following properties: (a) $R\neq 0$, (b) the transform $\bar{R}$ of R by (9), (10), (11) is given by

$$\bar{R} = \lambda^k AR/\Delta. \tag{14}$$

Now define the function p_σ by the expression

$$p_\sigma = f_\sigma + \frac{\partial}{\partial u^\sigma}\log R. \tag{15}$$

The point whose homogeneous projective coordinates are defined by the formula

$$r_\sigma = \frac{\partial x}{\partial u^\sigma} + p_\sigma x \tag{16}$$

transforms by (9), (10), (11) into $\bar{r}_{\bar{\sigma}}$ where

$$\bar{r}_{\bar{\sigma}} = \lambda\frac{\partial u^\rho}{\partial \bar{u}^\sigma}\left[\frac{\partial \bar{x}}{\partial u^\rho} + \left(p_\rho - \frac{\partial}{\partial u^\rho}\log\lambda\right)\bar{x}\right]. \tag{17}$$

Hence if we choose λ so that

$$\frac{\partial}{\partial u^\rho}\log\lambda = p_\rho,$$

we make

$$r_\sigma = \frac{\partial x}{\partial u^\sigma}, \tag{18}$$

and from (17)

$$\bar{r}_{\bar{\sigma}} = \lambda\frac{\partial u^\rho}{\partial \bar{u}^\sigma}r_\rho.$$

Hence r_σ is an intrinsic covariant vector.

The form of (18) of course depends upon the choice of the function R. In particular, the function

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y\right)$$

has the desired properties (a) and (b), and could be used in deriving (18). The function R may be chosen conveniently to the problem at hand. The coordinates x so normalized cause the points whose coordinates are $\partial x/\partial u^\rho$ *to lie in a covariant flat space* R_{k-1} of $k-1$ *dimensions lying in the tangent space* T_k.

Now if in the transformation (11) we choose ${}_{(s)}\theta^\rho$ and ${}_{(s)}\phi$ to satisfy the differential equations

$$
(19)\qquad
\begin{aligned}
\frac{\partial\ {}_{(s)}\theta^\sigma}{\partial u^\alpha} &= {}_{(s)}M^\sigma_\alpha - {}_{(s)}\theta^\rho L^\sigma_{\alpha\rho} - {}_{(s)}\phi\delta^\sigma_\alpha + \left({}_{(st)}E_\alpha - {}_{(s)}\theta^\rho\ {}_{(t)}D_{\alpha\rho}\right){}_{(t)}\theta^\sigma,\\
\frac{\partial\ {}_{(s)}\phi}{\partial u^\alpha} &= {}_{(s)}q_\alpha - {}_{(s)}\theta^\rho p_{\alpha\rho} + \left({}_{(st)}E_\alpha - {}_{(s)}\theta^\rho\ {}_{(t)}D_{\alpha\rho}\right){}_{(t)}\phi,
\end{aligned}
$$

whose integrability conditions appear among those of system (3), we cause (3) to assume a form for which

$$
{}_{(s)}M^\rho_\alpha = {}_{(s)}q_\alpha = 0, \qquad s = 1, 2, \cdots, r;\ \alpha, \rho = 1, 2, \cdots, k.
$$

Moreover, we may choose A of (11) so that ${}_{(ss)}E_\alpha = 0$. Hence by proper choice of the space N_r of points x, ${}_{(s)}y$ we may cause the system (3) to assume *a semi-canonical form*

$$
(20)\qquad
\begin{aligned}
\frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} &= L^\rho_{\alpha\beta}\frac{\partial x}{\partial u^\rho} + p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\ {}_{(s)}y,\\
\frac{\partial\ {}_{(s)}y}{\partial u^\alpha} &= {}_{(st)}E_\alpha\ {}_{(t)}y,
\end{aligned}
$$

characterized by

$$
L^\rho_{\rho\alpha} + \frac{\partial}{\partial u^\alpha}\log R = {}_{(ss)}E_\alpha = 0.
$$

The space N_r of r dimensions determined by x and ${}_{(s)}y$ giving rise to the form (20) may be called a *projective normal space*. Its precise nature depends upon the particular choice of the function R.

The form of (20) is preserved under all transformations of the form

$$
\begin{aligned}
u^\alpha &= u^\alpha(\bar u^1, \bar u^2, \cdots, \bar u^k), \qquad x = c\bar x, \qquad c \text{ a constant},\\
{}_{(s)}y &= a_{st}\ {}_{(t)}\bar y, \qquad\qquad\qquad\qquad\qquad A \text{ a constant}.
\end{aligned}
$$

5. **Covariant differentiation.** The tensor $p_{\alpha\beta}$ of the semi-canonical form could be used as a basis for an absolute calculus. We notice, however, that the functions $L^\rho_{\alpha\beta}$ transform under (9) by the same law of transformation as that by which the Christoffel symbols transform. Hence these functions can be used as a basis for covariant differentiation.

As defined* by Veblen and Thomas, "covariant differentiation is a process by which from a given tensor, there may be found a new tensor with one more covariant index." The definitions we use are the usual ones, and bear a close analogy to covariant differentiation in euclidean space S_3.

We denote covariant differentiation with respect to $L^\rho_{\alpha\beta}$ by a comma. In particular,

$$x_{,\alpha} = \frac{\partial x}{\partial u^\alpha}; \tag{21}$$

$$x_{,\alpha\beta} = \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} - L^\rho_{\alpha\beta}\frac{\partial x}{\partial u^\rho}. \tag{22}$$

But from (20), $x_{,\alpha\beta}$ may be written

$$x_{,\alpha\beta} = p_{\alpha\beta}x + {}_{(s)}D_{\alpha\beta}\,{}_{(s)}y. \tag{23}$$

Hence, *similar to euclidean geometry of n dimensions, the second covariant derivative of x gives a point (or a line through x) in the unique normal space.*

Now if one differentiates the equations (3) covariantly with respect to $L^\rho_{\alpha\beta}$, one finds the integrability conditions in covariant derivative form. These conditions are given by (5) wherein the comma is to be interpreted as the symbol of covariant differentiation. Using system (20) instead of (3), we may write the integrability conditions in the simple form

$$\begin{gathered} P^\rho_{\alpha\beta\gamma} = 0, \qquad p_{\alpha\beta,\gamma} - p_{\alpha\gamma,\beta} = 0, \\ {}_{(s)}D_{\alpha\beta,\gamma} - {}_{(s)}D_{\alpha\gamma,\beta} = {}_{(ts)}E_\beta\,{}_{(t)}D_{\alpha\gamma} - {}_{(ts)}E_\gamma\,{}_{(t)}D_{\alpha\beta}, \\ {}_{(st)}E_{\alpha,\beta} - {}_{(st)}E_{\beta,\alpha} = {}_{(sp)}E_\beta\,{}_{(pt)}E_\alpha - {}_{(sp)}E_\alpha\,{}_{(pt)}E_\beta. \end{gathered} \tag{24}$$

6. **Affine and "projective" connections.** The functions $L^\rho_{\alpha\beta}$ may be used to define an affine connection† on V_k since by (12) they transform according to the law of such a connection.‡ Since the functions $L^\rho_{\alpha\beta}$ are invariant under the projective transformation (4) in S_n, whatever may be valid for a given V_k in S_n is equally valid for any projective transform of V_k in S_n. We may therefore instigate a study of the geometry of paths on V_k from the point of view of classical differential geometry by the use of this connection.

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* Veblen and Thomas [**12**, p. 569].

† Such a connection is said to be of zero torsion, since $L^\rho_{\alpha\beta} = L^\rho_{\beta\alpha}$. The term "torsion" was introduced by Eddington [**13**].

‡ Schouten [**14**].

A curve on V_k will be said to be a path if the functions $u^\alpha = u^\alpha(t)$ defining the curve satisfy the differential equations

$$\frac{d^2u^\rho}{dt^2} + L^\rho_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt} = 0. \tag{25}$$

It may readily be proved that the osculating plane of the curve $u^\alpha = u^\alpha(v)$ at x intersects the space N_r in a line if and only if the functions $u^\alpha(v)$ satisfy equations of the form

$$\frac{d^2u^\rho}{dv^2} + L^\rho_{\alpha\beta}\frac{du^\alpha}{dv}\frac{du^\beta}{dv} = \theta\frac{du^\rho}{dv}, \qquad \theta \text{ arbitrary}. \tag{26}$$

But by proper choice of the function $t = t(v)$, we may write (26) in the form (25). Curves whose osculating planes intersect a given space N_r in lines are said to form an axial system.* In a sense, therefore, *the paths determined by the connection $L^\rho_{\alpha\beta}$ play the role of geodesics on a variety V_k in euclidean space S_n since the osculating planes of such geodesics intersect the unique normal space in lines.*

Consider now the so-called "projective" connection†

$$\Pi^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} - \frac{\delta^\rho_\alpha L^\sigma_{\sigma\beta}}{k+1} - \frac{\delta^\rho_\beta L^\sigma_{\sigma\alpha}}{k+1}. \tag{27}$$

In view of the conditions imposed on the differential equations (3) to reduce them to the form (20), we may write (27) in the form

$$\Pi^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} + \delta^\rho_\alpha\frac{\partial}{\partial u^\beta}\log R' + \delta^\rho_\beta\frac{\partial}{\partial u^\alpha}\log R', \tag{28}$$

wherein

$$R' = R^{-1/(k-1)}.$$

Hence $\Pi^\rho_{\alpha\beta}$ *defines the same set of paths*‡ *as* $L^\rho_{\alpha\beta}$. Therefore $\Pi^\rho_{\alpha\beta}$ *defines a projective connection in the sense of Thomas and, moreover, is projective in the sense of classical projective geometry.*

If we let $\mathfrak{B}^\rho_{\alpha\beta\gamma}$ be the curvature tensor for the connection $\Pi^\rho_{\alpha\beta}$, and let

$$r_{\alpha\beta} = \frac{\mathfrak{B}^\rho_{\alpha\beta\rho}}{k-1},$$

the Weyl projective curvature tensor $W^\rho_{\alpha\beta\gamma}$ is defined by

* Bortolotti [**15**].

† Thomas [**16**].

‡ Thomas [**16**] and Bortolotti [**17**].

$$(29)\qquad W^{\rho}_{\alpha\beta\gamma} = \mathfrak{B}^{\rho}_{\alpha\beta\gamma} + \delta^{\rho}_{\beta} r_{\alpha\gamma} - \delta^{\rho}_{\gamma} r_{\alpha\beta}.$$

We find that

$$(30)\quad W^{\rho}_{\alpha\beta\gamma} = R^{\rho}_{\alpha\beta\gamma} + \frac{\delta^{\rho}_{\beta} R^{\sigma}_{\alpha\gamma\sigma}}{k-1} - \frac{\delta^{\rho}_{\gamma} R^{\sigma}_{\alpha\beta\sigma}}{k-1} = P^{\rho}_{\alpha\beta\gamma} + \frac{\delta^{\rho}_{\beta} P^{\sigma}_{\alpha\gamma\sigma}}{k-1} - \frac{\delta^{\rho}_{\gamma} P^{\sigma}_{\alpha\beta\sigma}}{k-1}.$$

Hence

$$W^{\rho}_{\alpha\beta\gamma} = 0.$$

It follows, therefore, that the choice of the space N_r giving rise to the semi-canonical form (20) implies that the variety V_k, $(k>2)$, is projectively plane.* Moreover, for $k>2$ there is a preferred choice of parameters u^α such that $L^{\rho}_{\alpha\beta}$ are zero. Hence with $k>2$, and by proper choice of the transformation (10) we may cause system (20) to assume the form

$$(31)\qquad \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} = p_{\alpha\beta} x + {}_{(s)}D_{\alpha\beta}\, {}_{(s)}y, \qquad \frac{\partial\, {}_{(s)}y}{\partial u^\alpha} = {}_{(st)}E_\alpha\, {}_{(t)}y.$$

7. **Other connections on** V_k. By proper, but not unique, choice of the functions ${}_{(s)}\theta^\rho$, ${}_{(s)}\phi$ in transformation (11), and with certain limitations on the tensors ${}_{(s)}D_{\alpha\beta}$, we may make

$$(32)\qquad {}_{(ss)}E_\alpha = {}_{(t)}M^{\sigma}_{\sigma} = 0, \qquad \alpha = 1, 2, 3, \cdots, k;\ t = 1, 2, \cdots, r.$$

Under these conditions, the Weyl "projective" curvature tensor is still given by (30). Under the conditions (32) we may write (30) in the form

$$(33)\qquad W^{\rho}_{\alpha\beta\gamma} = {}_{(s)}D_{\alpha\sigma}\, {}_{(s)}Q^{\rho\sigma}_{\beta\gamma}$$

wherein

$${}_{(s)}Q^{\rho\sigma}_{\beta\gamma} = \delta^{\sigma}_{\beta}\, {}_{(s)}M^{\rho}_{\gamma} - \delta^{\sigma}_{\gamma}\, {}_{(s)}M^{\rho}_{\beta} + \frac{\delta^{\rho}_{\gamma}\, {}_{(s)}M^{\sigma}_{\beta}}{k-1} - \frac{\delta^{\rho}_{\beta}\, {}_{(s)}M^{\sigma}_{\gamma}}{k-1}.$$

The Weyl projective curvature tensor vanishes for $k=2$. Moreover, we may show that the geometry of paths under the conditions (32) possesses an invariant† integral of the form

$$(34)\qquad \int R^{-1} du^1 du^2 \cdots du^k.$$

Similarly, the geometry of paths, based on the conditions giving rise

* Bortolotti [**17**].

† Veblen [**18**]; Eisenhart [**19**].

to the semi-canonical form (20), also possesses an invariant integral.

For a general connection $L^{\rho}_{\alpha\beta}$ based on the defining system (3), the projective connection $\Pi^{\rho}_{\alpha\beta}$ transforms by (11) into $\overline{\Pi}^{\rho}_{\alpha\beta}$ by the formula

$$\overline{\Pi}^{\rho}_{\alpha\beta} = \Pi^{\rho}_{\alpha\beta} + {}_{(s)}\theta^{\sigma}\ {}_{(s)}Q^{\rho}_{\alpha\beta\sigma}, \tag{35}$$

wherein

$$ {}_{(s)}Q^{\rho}_{\alpha\beta\sigma} = \delta^{\rho}_{\sigma}\ {}_{(s)}D_{\alpha\beta} - \frac{\delta^{\rho}_{\alpha}\ {}_{(s)}D_{\sigma\beta}}{k+1} - \frac{\delta^{\rho}_{\beta}\ {}_{(s)}D_{\alpha\sigma}}{k+1}. \tag{36}$$

Hence the geometry of paths based on the projective connection $\Pi^{\rho}_{\alpha\beta}$ is independent of the space N_r if and only if the tensors ${}_{(s)}Q^{\rho}_{\alpha\beta\sigma}$ vanish; that is, if and only if the variety V_k is a linear space.

8. **The parallel displacement of Levi-Civita.** Consider given the contravariant components of a vector λ^{ρ} and a curve C with parametric equations $u^{\alpha}=u^{\alpha}(t)$ on V_k. If the components λ^{ρ} satisfy the differential equations

$$\frac{d\lambda^{\rho}}{dt} + L^{\rho}_{\alpha\beta}\lambda^{\alpha}\frac{du^{\beta}}{dt} = 0, \tag{37}$$

it is said that *the vector has suffered a parallel displacement along C.* If (37) is satisfied for all curves C through x on V_k, then

$$\lambda^{\rho}_{,\alpha} = \frac{\partial\lambda^{\rho}}{\partial u^{\alpha}} + L^{\rho}_{\alpha\sigma}\lambda^{\sigma} = 0, \tag{38}$$

and *it is said of the field of contravariant vectors that they are parallel with respect to any curve.*

Now consider any point z whose coordinates are determined by the expression

$$\nu z = \mu x + \lambda^{\rho}\frac{\partial x}{\partial u^{\rho}} + a_s\ {}_{(s)}y. \tag{39}$$

As x moves along a curve C_x $(u^{\alpha}=u^{\alpha}(t))$ on V_k, z moves along a curve C_z on a variety V_k'. The tangent to C_z at z is determined by z and by the point defined by the expression

$$\begin{aligned}\frac{d(\nu z)}{dt} &= \frac{d(\mu x)}{dt} + \frac{\partial x}{\partial u^{\rho}}\left[\frac{d\lambda^{\rho}}{dt} + L^{\rho}_{\alpha\beta}\lambda^{\alpha}\frac{du^{\beta}}{dt}\right] \\ &\quad + \left[p_{\rho\beta}\lambda^{\rho}x + \left(\frac{\partial a_s}{\partial u^{\beta}} + {}_{(ts)}E_{\beta}a_t + \lambda^{\alpha}\ {}_{(s)}D_{\alpha\beta}\right){}_{(s)}y\right]\frac{du^{\beta}}{dt}.\end{aligned} \tag{40}$$

Hence the tangent to C_z at z and the tangent to C_x at x are coplanar if and only if

$$\text{(41)}\qquad \begin{aligned} \frac{d\lambda^\rho}{dt} + L^{\rho}_{\alpha\beta}\lambda^{\alpha}\frac{du^\beta}{dt} &= 0, \\ \frac{da_s}{dt} + ({}_{(ts)}E_\beta a_t + \lambda^\alpha {}_{(s)}D_{\alpha\beta})\frac{du^\beta}{dt} &= 0. \end{aligned}$$

If equations (41) hold, we may say that C_x and C_z are in relation C, *a sort of generalization of the transformation of Combescure.* If (41) is to hold for all curves on V_k, then λ^ρ and a_s must satisfy the equations

$$\text{(42)}\qquad \begin{aligned} \lambda^{\rho}_{,\alpha} = \frac{\partial\lambda^\rho}{\partial u^\alpha} + L^{\rho}_{\sigma\alpha}\lambda^{\sigma} &= 0, \\ \frac{\partial a_s}{\partial u^\alpha} + {}_{(ts)}E_\alpha a_t + \lambda^\sigma {}_{(s)}D_{\alpha\sigma} &= 0. \end{aligned}$$

The tangents to any curve on V_k intersect the tangents to the corresponding curves on V_k'. We may say that V_k and V_k' are *in relation C.* Granted that the first of equations* (42) has solutions, the integrability conditions of the last of (42) follow from those of system (3).

The first of (42) has the "trivial" solution $\lambda^\rho = 0$. Then the tangents to the corresponding curves on V_k and V_k' are coplanar if and only if a_s satisfies the equations

$$\text{(43)}\qquad \frac{\partial a_s}{\partial u^\alpha} + {}_{(ts)}E_\alpha a_t = 0.$$

The integrability conditions of (43) follow from those of (3). But *if* (43) *is satisfied, the line xz passes through a fixed point.* We shall say that V_k and V_k' are *in the relation of a radial transformation.* If $k=2$, $r=1$, and the varieties V_k and V_k' are ordinary surfaces in S_3, then *the two surfaces are in the relation of a radial transformation* as the term is usually understood.

9. **Conclusion.** Finally, in conclusion, let there be given a generalized Riemann space† S_k the coordinates of whose points are $u^1, u^2, \cdots, u^k$. Let there be given a fundamental affine connection $\Lambda^{\rho}_{\alpha\beta}$ determining the structure of the space. Suppose, moreover, that $\Lambda^{\rho}_{\alpha\beta}$ is a symmetric connection and that under the transformation

$$u^\alpha = u^\alpha(\bar{u}^1, \bar{u}^2, \cdots, \bar{u}^k)$$

* Meyer and Thomas [20].

† Thomas [21]; Hlavatý [22].

the functions $\Lambda^\rho_{\alpha\beta}$ suffer the usual transformation

$$\bar{\Lambda}^\rho_{\alpha\beta} = \left[\Lambda^\lambda_{\gamma\delta} \frac{\partial u^\gamma}{\partial \bar{u}^\alpha} \frac{\partial u^\delta}{\partial \bar{u}^\beta} + \frac{\partial^2 u^\lambda}{\partial \bar{u}^\alpha \partial \bar{u}^\beta}\right] \frac{\partial \bar{u}^\rho}{\partial u^\lambda} \tag{44}$$

of such connections. The structure of the space S_k is equally as well determined by a system of paths on S_k: $u^\alpha = u^\alpha(t)$, where the functions $u^\alpha(t)$ satisfy the usual differential equations of such paths, namely

$$\frac{d^2 u^\rho}{dt^2} + \Lambda^\rho_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt} = 0. \tag{45}$$

Let the space S_k be mapped on a variety V_k immersed in a projective space S_n of $n = k(k+3)/2$ dimensions. Then the $n+1$ homogeneous projective coordinates of a point P on V_k are expressible as functions of $u^1, u^2, \cdots, u^k$. Let ${}_{(s)}y$ be the homogeneous projective coordinates of $r = n(n+1)/2$ other points in S_n such that

$$R = \left(x, \frac{\partial x}{\partial u^1}, \frac{\partial x}{\partial u^2}, \cdots, \frac{\partial x}{\partial u^k}, {}_{(1)}y, {}_{(2)}y, \cdots, {}_{(r)}y\right) \neq 0.$$

Then the functions x and ${}_{(s)}y$ satisfy a system of equations of the form (3).

By a transformation of the form (11) we may make

$$L^\rho_{\alpha\beta} + {}_{(s)}\theta^\rho \, {}_{(s)}D_{\alpha\beta} = \Lambda^\rho_{\alpha\beta}, \tag{46}$$

since the rank of the matrix $({}_{(s)}D_{\alpha\beta})$ is r. Hence we may cause equations (3) to assume the form

$$\begin{aligned} \frac{\partial^2 x}{\partial u^\alpha \partial u^\beta} &= \Lambda^\rho_{\alpha\beta} \frac{\partial x}{\partial u^\rho} + p_{\alpha\beta} x + {}_{(s)}D_{\alpha\beta} \, {}_{(s)}y, \\ \frac{\partial \, {}_{(s)}y}{\partial u^\alpha} &= {}_{(s)}M^\rho_\alpha \frac{\partial x}{\partial u^\rho} + {}_{(s)}q_\alpha x + {}_{(st)}E_\alpha \, {}_{(t)}y. \end{aligned} \tag{47}$$

With Thomas we may describe the *"projective" theory of connections as the study of all possible affine connections*. From this point of view *the study of equations* (46) *for all possible choices of the space N_r of points x, ${}_{(s)}y$* (that is, of all possible choices of ${}_{(s)}\theta^\rho$) *is the projective study of connections in the symmetric case.*

Moreover, under the transformation (10) the coefficients $\bar{\Lambda}^\rho_{\alpha\beta}$ transform by (10) according to the formula

$$\bar{\Lambda}^\rho_{\alpha\beta} = \Lambda^\rho_{\alpha\beta} - \delta^\rho_\alpha \frac{\partial}{\partial u^\beta} \log \lambda - \delta^\rho_\beta \frac{\partial}{\partial u^\alpha} \log \lambda.$$

Hence* $\overline{\Lambda}^{\rho}_{\alpha\beta}$ *defines on* S_k *the same system of paths as do* $\Lambda^{\rho}_{\alpha\beta}$. We may state our result in the following form:

Let the space S_k *of structure* $\Lambda^{\rho}_{\alpha\beta}$ *be mapped on a variety* V_k *in a projective space* S_n *of* $n=k(k+3)/2$ *dimensions. Then the paths of the space* S_k *map into the curves of the variety* V_k *whose osculating planes intersect an unique space of points* N_r *of* $r=k(k+1)/2$ *dimensions in lines.* In other words, *a generalized Riemann space with a given structure may be considered as immersed in a projective space of sufficiently high dimension.*

In particular, if we first normalize the coordinates x by a transformation (10), and then choose the space N_r of points x, ${}_{(s)}y$, we may make the connection $L^{\rho}_{\alpha\beta}$ of (3) assume the form $\{\rho, \alpha\beta\}$ of the Christoffel symbols obtained from a so-called fundamental metric tensor $g_{\alpha\beta}$ of a Riemann space S_k. *The space* N_r *would then be uniquely determined by the metric tensor, and any projective transform of* V_k *in* S_n *would possess the same Riemannian metric.* The geodesics from this Riemann geometric point of view would be the curves on V_k whose osculating planes intersect an unique normal space N_r in lines just as occurs in the euclidean geometry of a variety V_k in S_n.

References

1. E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932.

2. L. P. Eisenhart, *Continuous Groups of Transformations*, Princeton University Press, 1933.

3. E. P. Lane, *The projective differential geometry of systems of linear differential equations of the first order*, Transactions of this Society, vol. 30 (1928), pp. 785–796.

4. Fubini and Čech, *Geometria Proiettiva Differenziale*, vol. 1, 1926; vol. 2, 1927, Bologna, Zanichelli.

5. ———, *Introduction à la Géométrie Projective Différentielle des Surfaces*, Paris, Gauthier-Villars, 1931.

6. V. Hlavatý, *Invariants projectifs d'une hypersurface*, Rendiconti del Circolo Matematico di Palermo, vol. 57 (1933), pp. 402–432.

7. E J. Wilczynski, *Projective Differential Geometry*, Leipzig, Teubner, 1906.

8. ———, *Projective differential geometry of curved surfaces (first memoir)*, Transactions of this Society, vol. 8 (1907), pp. 233–260.

9. ———, *Sur la théorie générale des congruences*, Mémoires de la Classe des Sciences de l'Académie Royale de Belgiques, (2), vol. 3 (1911).

10. E. P. Lane, *A canonical power series expansion for a surface*, Transactions of this Society, vol. 37 (1935), pp. 463–482.

11. V. G. Grove, *On canonical forms of differential equations*, this Bulletin, vol. 36 (1930), pp. 582–586.

* Weyl [**23**]; Thomas [**21**].

12. Veblen and Thomas, *The geometry of paths*, Transactions of this Society, vol. 25 (1923), pp. 551, 608.

13. A. S. Eddington, *A generalization of Weyl's theory of electromagnetism and gravitational fields*, Proceedings of the Royal Society, vol. 99 (1921), pp. 104–122.

14. J. A. Schouten, *Der Ricci-Kalkül*, Springer, 1924.

15. E. Bortolotti, *Sistemi assaile e connessione nelle V_n*, Rendiconti dell'Accademia dei Lincei, (6), vol. 5 (1927), pp. 390–395.

16. T. Y. Thomas, *On the projective and equi-projective geometry of paths*, Proceedings of the National Academy of Sciences, vol. 2 (1926).

17. E. Bortolotti, *Spazi proiettivamenti piani*, Annali di Matematica, vol. 2 (1932–1933), pp. 111–134.

18. O. Veblen, *Equi-affine geometry of paths*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 3–4.

19. L. P. Eisenhart, *Affine geometry of paths possessing an invariant integral*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 4–7.

20. Meyer and Thomas, *Fields of parallel vectors in non-analytic manifolds in the large*, Composito Mathematica, vol. 5 (1938), p. 126.

21. T. Y. Thomas, *Recent trends in geometry*, *Semicentennial Addresses of the American Mathematical Society*, vol. 2, 1938, pp. 98–135.

22. V. Hlavatý, *Connexion projective et déplacement projectif*, Annali di Matematica, (4), vol. 12 (1934), pp. 217–294.

23. H. Weyl, *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1921, pp. 99–112.

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