# THE STIELTJES MOMENT PROBLEM FOR FUNCTIONS OF BOUNDED VARIATION

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1. **Introduction.** We shall establish the following theorem, which at first sight appears quite unexpected:

THEOREM 1. *Any sequence* $\{\mu_n\}$ *of real numbers can be represented in the form*

$$(1.1)\qquad \mu_n = \int_0^\infty t^n d\alpha(t), \qquad n = 0, 1, 2, \cdots, \qquad \int_0^\infty |d\alpha(t)| < \infty.$$

The problem of determining necessary and sufficient conditions for a sequence of numbers $\{\mu_n\}$ to have the form

$$(1.2)\qquad \mu_n = \int_0^\infty t^n d\alpha(t), \qquad \alpha(t) \text{ non-decreasing}, n = 0, 1, 2, \cdots,$$

was set and solved by T. J. Stieltjes. It would be natural to attempt to generalize the problem by requiring merely that $\alpha(t)$ should be a function of bounded variation on $(0, \infty)$; but the generalized problem has, as Theorem 1 shows, a trivial solution.

To establish Theorem 1, we shall exhibit an arbitrary real sequence $\{\mu_n\}$ as the difference of two sequences $\{\lambda_n\}$ and $\{\nu_n\}$, each of the form (1.2).† The construction will also lead to the result that any sequence $\{\mu^n\}$ of positive numbers of sufficiently rapid growth has the form (1.2); it is sufficient, for example, that

$$(1.3)\qquad \mu_0 \geqq 1, \qquad \mu_n \geqq (n\mu_{n-1})^n, \qquad n \geqq 1.$$

A specimen sequence satisfying (1.3) is $\mu_0 = 1, \mu_n = n^{n^n}$, $(n = 1, 2, \cdots)$.

As an application‡ of Theorem 1, it will be shown that

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† Added in proof: Other proofs of Theorem 1 have been given by G. Pólya (*Sur l'indétermination d'un problème voisin du problème des moments*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 708–711). Pólya points out that a theorem of which Theorem 1 is an immediate consequence was proved by É. Borel in 1894.

‡ For another application of Theorem 1, see J. Shohat, *Sur les polynomes orthogonaux généralisés*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 207 (1938), pp 556–558.

$$f(x) = \int_0^\infty x(t)d\alpha(t),$$

with

$$\int_0^\infty t^n \left| d\alpha(t) \right| < \infty, \qquad n = 1, 2, \cdots,$$

is not the general linear functional on any very interesting space of functions $x=x(t)$, containing an infinite number of the functions $t^n$, $(n=1, 2, \cdots)$ (see §4 for a precise statement). Other negative results of this character have been obtained by J. W. Tukey and the author;* the reader is referred to their paper for a discussion of the significance of such results.

2. **Proof of Theorem** 1. We use the notation

$$[\mu_0\mu_2 \cdots \mu_{2n}] = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \cdots.$$

Then a necessary and sufficient condition for $\{\mu_n\}$ to have the form 1.2 is†

$$(2.1) \qquad [\mu_0\mu_2 \cdots \mu_{2n}] \geqq 0, \quad [\mu_1\mu_3 \cdots \mu_{2n+1}] \geqq 0, \quad n = 0, 1, 2, \cdots.$$

We choose positive numbers $\lambda_0$, $\lambda_1$, $\nu_0$, $\nu_1$, so that $\lambda_0-\nu_0=\mu_0$, $\lambda_1-\nu_1=\mu_1$. We now proceed to define the sequences $\{\lambda_n\}$, $\{\nu_n\}$ by induction. Suppose that

$$(2.2) \qquad \lambda_k - \nu_k = \mu_k$$

for $k=0, 1, 2, \cdots, 2n-1$, and that the determinants

$$(2.3) \qquad \begin{matrix} [\lambda_0\lambda_2 \cdots \lambda_{2k}], & [\nu_0\nu_2 \cdots \nu_{2k}], \\ [\lambda_1\lambda_3 \cdots \lambda_{2k+1}], & [\nu_1\nu_3 \cdots \nu_{2k+1}], \end{matrix}$$

are positive for $k=0, 1, 2, \cdots, n-1$. We have (with undetermined $\lambda_{2n}$)

$$[\lambda_0\lambda_2 \cdots \lambda_{2n}] = \lambda_{2n}[\lambda_0\lambda_2 \cdots \lambda_{2n-2}] + P,$$

where $P$ is a polynomial in $\lambda_0, \lambda_1, \cdots, \lambda_{2n-1}$; and there is a corre-

* R. P. Boas, Jr., and J. W. Tukey, *A note on linear functionals*, this Bulletin, vol. 44 (1938), pp. 523–528.

† See, for example, O. Perron, *Die Lehre von den Kettenbrüchen*, 1929, p. 410; cf. also M. Riesz, *Sur le problème des moments, troisième note*, Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1922–1923), no. 16.

sponding relation for $[\nu_0\nu_2 \cdots \nu_{2n}]$. Since $[\lambda_0\lambda_2 \cdots \lambda_{2n-2}]>0$, and $[\nu_0\nu_2 \cdots \nu_{2n-2}]>0$, we can choose $\lambda_{2n}$ and $\nu_{2n}$ so that $\lambda_{2n}-\nu_{2n}=\mu_{2n}$, and so large that $[\lambda_0\lambda_2 \cdots \lambda_{2n}]>0$, $[\nu_0\nu_2 \cdots \nu_{2n}]>0$. Similarly we can then choose $\lambda_{2n+1}$ and $\nu_{2n+1}$ so that $\lambda_{2n+1}-\nu_{2n+1}=\mu_{2n+1}$, $[\lambda_1\lambda_3 \cdots \lambda_{2n+1}]>0$, $[\nu_1\nu_3 \cdots \nu_{2n+1}]>0$. This completes the induction: we can find sequences $\{\lambda_n\}$, $\{\nu_n\}$ such that for $k=0, 1, 2, \cdots$, (2.2) is satisfied, and all the determinants (2.3) are positive. Then $\{\lambda_n\}$ and $\{\nu_n\}$ satisfy (2.1), and consequently have the form (1.2), so that $\{\mu_n\}$ has the form (1.1).

3. **Rapidly increasing sequences.** We now prove the following theorem:

THEOREM 2. *If*

$$\mu_0 \geqq 1, \qquad \mu_n \geqq (n\mu_{n-1})^n, \qquad n = 1, 2, \cdots, \tag{3.1}$$

*then* $\{\mu_n\}$ *has the form* (1.2).

For the proof, we modify the construction of the sequence $\{\lambda_n\}$ of §2. We have, for $n=1, 2, \cdots$,

$$[\mu_0\mu_2 \cdots \mu_{2n}] = \mu_{2n}[\mu_0\mu_2 \cdots \mu_{2n-2}] + \sum_{k=n}^{2n-1} \pm \mu_k D_k, \tag{3.2}$$

where the $D_k$ are $n$-rowed minors of $[\mu_0\mu_2 \cdots \mu_{2n}]$ and do not involve $\mu_{2n}$. Similarly, for $n=1, 2, \cdots$,

$$[\mu_1\mu_3 \cdots \mu_{2n+1}] = \mu_{2n+1}[\mu_1\mu_3 \cdots \mu_{2n-1}] + \sum_{k=n+1}^{2n} \pm \mu_k D_k', \tag{3.3}$$

where the $D_k'$ are $n$-rowed minors of $[\mu_1\mu_3 \cdots \mu_{2n+1}]$, not involving $\mu_{2n+1}$.

Suppose that for $k \leqq n-1$, $(n=1, 2, \cdots)$,

$$[\mu_0\mu_2 \cdots \mu_{2k}] \geqq 1, \qquad [\mu_1\mu_3 \cdots \mu_{2k+1}] \geqq 1. \tag{3.4}$$

Assuming (3.1), we shall show that (3.4) is satisfied also for $k=n$.

Clearly, $\mu_m \geqq 1$ for $m=1, 2, \cdots$. Hence we have

$$\mu_m \geqq (m\mu_{m-1})^m > 2(m/2)^{(m+4)/4}\mu_{m-1}^{(m+2)/2}, \quad m = 2, 3, \cdots.$$

Therefore

$$\mu_{2n} > 1 + n^{(n+2)/2}\mu_{2n-1}^{n+1}, \qquad \mu_{2n+1} > 1 + n^{(n+2)/2}\mu_{2n}^{n+1}. \tag{3.5}$$

Now, (3.1) implies in particular that $\mu_{m+1} \geqq \mu_m$, $(m \geqq 1)$; hence the elements of the determinants $D_k$ do not exceed $\mu_{2n-1}$, and the elements of the $D_k'$ do not exceed $\mu_{2n}$. Then by Hadamard's theorem,*

$$
\begin{aligned}
\left| D_k \right| &\leqq \mu_{2n-1}^{n} n^{n/2}, && k = n, n+1, \cdots, 2n-1,\\
\left| D_k' \right| &\leqq \mu_{2n}^{n} n^{n/2}, && k = n+1, n+2, \cdots, 2n.
\end{aligned}
$$

Therefore, using (3.2), (3.3), (3.4), (3.5), we obtain

$$
\begin{aligned}
[\mu_0\mu_2 \cdots \mu_{2n}] &\geqq \mu_{2n} - n^{1+n/2}\mu_{2n-1}^{n+1} > 1,\\
[\mu_1\mu_3 \cdots \mu_{2n+1}] &\geqq \mu_{2n+1} - n^{1+n/2}\mu_{2n}^{n+1} > 1.
\end{aligned}
$$

Thus (3.4) holds for $k=n$ if it holds for $k<n$; but it holds for $k=0$ by assumption, and consequently holds for all $k$; therefore $\{\mu_n\}$ has the form (1.2).

The moment problem (1.2) is said to be determined or undetermined according as the function $\alpha(t)$ is or is not unique (after being normalized by the conditions $\alpha(0)=0$, $\alpha(t)=[\alpha(t+)+\alpha(t-)]/2$ for $t>0$). A consequence of Theorem 2 is that the moment problem (1.2) is not only solvable for any sequence $\{\mu_n\}$ of sufficiently rapid growth, but is even undetermined. In fact, if $\{\mu_n\}$ satisfies (3.1) and if in addition $\mu_2 \geqq (2\mu_1+2)^2$, we define a sequence $\{\nu_n\}$ by setting $\nu_1=\mu_1+1$, $\nu_n=\mu_n$ for $n\neq 1$. Then $\{\nu_n\}$ satisfies (3.1); consequently for $n=0, 1, 2, \cdots$,

$$
\nu_{2n} = \int_0^\infty t^{2n} d\beta(t) = \int_0^\infty u^n d\beta(u^{1/2}) = \int_0^\infty u^n d\gamma(u),
$$

say; while

$$
\nu_{2n} = \mu_{2n} = \int_0^\infty t^{2n} d\alpha(t) = \int_0^\infty u^n d\delta(u),
$$

where $\gamma(u)$ and $\delta(u)$ are normalized and non-decreasing. But $\gamma(u)$ and $\delta(u)$ are distinct, since

$$
\nu_1 = \int_0^\infty u^{1/2} d\gamma(u) = 1 + \int_0^\infty u^{1/2} d\delta(u) = 1 + \mu_1.
$$

Hence the moment problem for the sequence $\{\mu_{2n}\}$ is undetermined.

4. **Linear functionals.** We use the terminology of S. Banach's book.† Let $R$ be a topological vector space of elements $x$, let $P$ be a

* G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, **1934, p. 34.**

† *Théorie des Opérations Linéaires*, 1932.

space of elements $p$, and let $f_p(x)$ be a functional with domain $R$, defined for each $p$ in $P$. We say that a general linear functional in $R$ is $f_p(x)$, if the following conditions are satisfied:

(i) *$f_p(x)$ is a linear functional for every $p \in P$.*

(ii) *Every linear functional $g(x)$ with domain $R$ is identically equal to some $f_p(x)$.*

In the application to be made here, the elements of $P$ are the functions $p=p(t)$, of bounded variation on $(0, \infty)$, such that

$$\int_0^\infty t^n \left|dp(t)\right| < \infty, \qquad n=1, 2, \cdots;$$

the elements of $R$ are measurable functions $x=x(t)$, defined on $(0, \infty)$; and

$$f_p(x) = \int_0^\infty x(t)dp(t), \tag{4.1}$$

where the integral is a Lebesgue-Stieltjes integral. We have the following result:

THEOREM 3. *Let $R$ be a topological vector space with the following property:**

*(Q): If $x \in R$ and $a_n \rightarrow 0$, then $a_n x \rightarrow \Theta$.†*

*Then if $R$ contains an infinite number of functions $t^n$, $(n=0, 1, 2, \cdots)$, there is some $p \in P$ for which* (4.1) *is not a linear functional on $R$.*

In particular, we see that, under the hypotheses of Theorem 3, (4.1) is not a general linear functional on $R$.

Suppose that (4.1) is, for every $p \in P$, a linear functional on a space $R$ with the specified properties. Let $S$ be the subspace composed of all finite linear combinations of the elements $t^n$ which are in $R$ (with the topology of $R$). If $f$ is an arbitrary distributive (that is, additive and homogeneous) functional with domain $S$, we define a sequence $\{\mu_n\}$ by setting $\mu_n = f(t^n)$ when $t^n \in R$, and $\mu_n = 0$ otherwise. By Theorem 1, there is a $p \in P$ such that

$$\mu_n = \int_0^\infty t^n dp(t), \qquad n = 0, 1, 2, \cdots.$$

Since $f$ is distributive, we then have

* In particular, a space of type $F$ has this property.

† $\Theta$ denotes the zero element of $R$.

$$f(x) = \int_0^\infty x(t)dp(t), \qquad x \varepsilon S. \tag{4.2}$$

Now (4.1) is a linear functional on $R$, and consequently a linear functional on $S$. Hence (4.2) states that every distributive functional on $S$ is linear; but this is impossible unless $S$ is finite-dimensional,* which it is not. This contradiction establishes the theorem.

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# ON FUNDAMENTAL SYSTEMS OF SYMMETRIC FUNCTIONS†

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A set $S$ of $n$ polynomials over a field $K$, symmetric in $n$ variables, $x_1, x_2, \cdots, x_n$, is said to form a fundamental system if any rational function over $K$, symmetric in these variables, can be expressed rationally in terms of the polynomials of $S$. In this paper we show that any $n$ algebraically independent symmetric polynomials over a field $K$ of characteristic zero form a fundamental system if the product of their degrees is less than $2n!$.

The result follows from a theorem due to Perron:‡

THEOREM 1. *Between $n+1$ polynomials (not constant), $f_1, f_2, \cdots, f_{n+1}$, in $n$ variables, of degrees $m_1, m_2, \cdots, m_{n+1}$, respectively, there is always an identity of the form*

$$\sum C_{\nu_1\nu_2\cdots\nu_{n+1}} f_1^{\nu_1} f_2^{\nu_2} \cdots f_{n+1}^{\nu_{n+1}} \equiv 0,$$

*where in each term,*

$$\sum_{i=1}^{n+1} m_i\nu_i \leqq \prod_{i=1}^{n+1} m_i.$$

* Let every distributive functional on $S$ be linear, where $S$ is a topological vector space with the property (Q). If $S$ is infinite dimensional, let $\{x_n\}$, $(n=1, 2, \cdots)$, be an infinite set of linearly independent elements. Since $\lim_{k\to\infty} k^{-1}x_n = \Theta$, we can choose $y_n \varepsilon S$, $(n=1, 2, \cdots)$, linearly independent, with $y_n \to \Theta$. We set $f(y_n)=1$, $f(x)=0$ when $x$ is not a finite linear combination of the $y_n$, $f(ax+by)=af(x)+bf(y)$ for any $x \varepsilon S$, $y \varepsilon S$; then $f$ is a distributive functional on $S$, and hence is linear on $S$. Since $y_n \to \Theta$, $f(y_n) \to 0$ as $n \to \infty$; but this contradicts $f(y_n)=1$. Consequently $S$ is finite dimensional.

† Presented to the Society, February 25, 1939, under the title *A note on fundamental systems of symmetric functions.*

‡ O. Perron, *Bemerkung zur Algebra*, Sitzungsberichte der Bayerischen Akademie, mathematisch-naturwissenschaftliche Abteilung, 1924, pp. 87–101.