A NOTE ON THE WEIERSTRASS CONDITION IN THE CALCULUS OF VARIATIONS*

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The present note is concerned with the proof of a simple property of the Weierstrass E-function which, so far as the authors know, has not been pointed out before. For the sake of generality, the result will be given for the general problem of the Lagrange or Bolza type. The result for such a problem in non-parametric form is given in Theorem I below, and the analogous result for the parametric problem is presented in Theorem II.

For the non-parametric problem let $f(x, y_1, \cdots, y_n, p_1, \cdots, p_n) = f(x, y, p)$ denote the integrand function and $\phi_\alpha(x, y, p)$, $(\alpha = 1, \cdots, m < n)$, the auxiliary expressions.† It will be assumed that the functions f, ϕ_α are continuous and have continuous derivatives of the first two orders in a region‡ $\mathcal{R}$ of (x, y, p)-space. By an *admissible set*

$$(x, y_1, \cdots, y_n, p_1, \cdots, p_n, \lambda_1, \cdots, \lambda_m) = (x, y, p, \lambda)$$

will be meant one such that (x, y, p) is in $\mathcal{R}$ and satisfies the equations $\phi_\alpha = 0$. Let $F(x, y, p, \lambda) = f(x, y, p) + \lambda_\alpha \phi_\alpha(x, y, p)$. Here and elsewhere in this note the tensor analysis summation convention is used.

THEOREM I. *Suppose N is a region in (x, y, p, λ)-space such that at each admissible set (x, y, p, λ) of N the inequality*

$$\begin{aligned} E(x, y, p, \lambda, q) \equiv F(x, y, q, \lambda) &- F(x, y, p, \lambda) \\ &- (q_i - p_i) F_{p_i}(x, y, p, \lambda) \geqq 0 \end{aligned} \tag{1}$$

holds for every set (q_i) for which (x, y, q, λ) is admissible. If the matrix

$$\left\| \begin{matrix} F_{p_i p_j} & \phi_{\beta p_i} \\ \phi_{\alpha p_j} & 0 \cdot \delta_{\alpha\beta} \end{matrix} \right\|, \qquad i, j = 1, \cdots, n; \alpha, \beta = 1, \cdots, m, \tag{2}$$

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† For a more detailed formulation of the problems of Lagrange and Bolza the reader is referred, for example, to Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673–742, Morse, *Sufficient conditions in the problem of Lagrange with variable end-conditions*, American Journal of Mathematics, vol. 53 (1931), pp. 517–546, or Bliss, *The Problem of Bolza in the Calculus of Variations*, mimeographed notes of lectures delivered winter, 1935, at the University of Chicago.

‡ By "region" we shall understand "open region." It is to be noted that in the following theorems no use is made of the region's being open with respect to the (x, y) or (y) variables. Consequently, the hypotheses of the theorems could be weakened in this respect.

is nonsingular at each admissible set (x, y, p, λ) *in* N, *the equality in* (1) *holds only in case* $p_i = q_i$, $(i=1, \cdots, n)$.

For suppose that the equality in (1) holds for a particular admissible set $(x_0, y_0, p_0, \lambda_0)$ of N and a set q_0 with $(x_0, y_0, q_0, \lambda_0)$ admissible. It would then follow that (p_0, λ_0) affords $E(x_0, y_0, p, \lambda, q_0)$ a minimum relative to neighboring sets (p, λ) for which $\phi_\alpha(x_0, y_0, p) = 0$, $(\alpha = 1, \cdots, m)$. Consequently, by the Lagrange multiplier rule,* there would exist multipliers l_α such that at $(x_0, y_0, p_0, \lambda_0, q_0)$ we have

$$E_{p_j} + l_\alpha \phi_{\alpha p_j} = 0, \qquad E_{\lambda_\alpha} = 0, \qquad j = 1, \cdots, n; \alpha = 1, \cdots, m;$$

that is,

$$-(q_{i0} - p_{i0}) F_{p_i p_j} + l_\alpha \phi_{\alpha p_j} = 0, \quad -(q_{i0} - p_{i0}) \phi_{\alpha p_i} = 0. \tag{3}$$

But in view of the nonsingularity of the matrix (2) at admissible sets in N, these equations imply $q_{i0} = p_{i0}$, $l_\alpha = 0$; hence the theorem is established.

For the problem of the calculus of variations in parametric form, the functions f, ϕ_α, $(\alpha = 1, \cdots, m < n-1)$, are assumed to be independent of the variable x and to be positively homogeneous of degree one in the variables p_i. It is also assumed that these functions are continuous and have continuous derivatives of the first two orders in a region $\mathcal{R}$ of (y, p)-space which is such that if (y, p) is in $\mathcal{R}$ then $p_i p_i \neq 0$, and, moreover, the point (y, kp), $(k > 0)$, is also in $\mathcal{R}$. An admissible set (y, p, λ) is defined in a manner analogous to that used above. One has the well known relations

$$\begin{gathered} F(y, p, \lambda) = p_i F_{p_i}(y, p, \lambda), \qquad \phi_\alpha(y, p) = p_i \phi_{\alpha p_i}(y, p), \\ p_i F_{p_i p_j}(y, p, \lambda) = 0, \end{gathered} \tag{4}$$

$$\begin{gathered} F_{p_i}(y, kp, \lambda) = F_{p_i}(y, p, \lambda), \\ k > 0; i, j = 1, \cdots, n; \alpha = 1, \cdots, m. \end{gathered} \tag{5}$$

From the second and third relations of (4) it follows that at an admissible set (y, p, λ) the matrix (2) is singular. It is also a consequence of (5) and the first equation of (4) that

$$E(y, p, \lambda, q) = q_i [F_{p_i}(y, q, \lambda) - F_{p_i}(y, p, \lambda)], \tag{6}$$

$$E(y, kp, \lambda, k'q) = k' E(y, p, \lambda, q), \qquad k > 0, k' > 0. \tag{7}$$

THEOREM II. *Suppose* N *is a region in* (y, p, λ)-*space such that at each admissible set* (y, p, λ) *of* N *the inequality*

* See, for example, Carathéodory, *Variationsrechnung*, p. 116.

$$E(y, p, \lambda, q) \geqq 0 \tag{8}$$

holds at every set (q_i) *for which* (y, q, λ) *is admissible. If the matrix* (2) *is of rank* $n+m-1$ *at each admissible set* (y, p, λ) *in* N, *the equality in* (8) *holds only in case* $q_i = hp_i$, $(h>0)$.

Suppose the equality in (8) holds for a particular admissible set (y_0, p_0, λ_0) of N and a set q_0 with (y_0, q_0, λ_0) admissible. In view of (7) we may assume that $p_{i0}p_{i0}=1$, $q_{i0}q_{i0}=1$. Proceeding as in the proof of Theorem I, we find that at the set $(y_0, p_0, \lambda_0, q_0)$ equations (3) hold. By virtue of relations (4) and the fact that the matrix (2) is of rank $n+m-1$ at (y_0, p_0, λ_0), equations (3) imply $q_{i0} = hp_{i0}$, $l_\alpha = 0$; moreover, since $p_{i0}p_{i0}=1$, $q_{i0}q_{i0}=1$, we have $h = \pm 1$. The inequality in (8) holds, therefore, for every set (y, p, λ, q) satisfying $p_ip_i=1$, $q_iq_i=1$, $(q_i) \neq \pm(p_i)$, (y, p, λ) an admissible set in N, (y, q, λ) an admissible set. But from the form (6) of $E(y, p, \lambda, q)$ one readily verifies that if $(y_0, -p_0)$ is in $\mathfrak{R}$, then

$$E(y_0, p_0, \lambda_0, -p_0) = E(y_0, r, \lambda_0, -p_0) + E(y_0, r, \lambda_0, p_0) \tag{9}$$

for every set r such that (y, r) is in $\mathfrak{R}$. In particular, since (y_0, p_0, λ_0) is an admissible set in N, it follows from the usual implicit function theorem that there exists a neighboring admissible set (y_0, r, λ_0) in N such that $r_ir_i=1$, $(r_i) \neq \pm(p_{i0})$. When this value of r is substituted in (9), it is found that $E(y_0, p_0, \lambda_0, -p_0) > 0$ if $(y_0, -p_0, \lambda_0)$ is admissible. Consequently, whenever (y, p, λ) is an admissible set in N, (y, q, λ) is an admissible set, and $p_ip_i=1$, $q_iq_i=1$, the equality in (8) holds if and only if $p_i = q_i$. From relation (7) this result is readily seen to be equivalent to the conclusion of Theorem II.

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