I			II		
(1,2,1,3),	(2,3,2,1),	(3,1,3,2),	(1,2,1,3),	(2,1,3,1),	(3,3,2,2),
(1,2,1,3),	(2,1,2,2),	(3,3,3,1),	(1,2,1,3),	(2,1,3,2),	(3,3,2,1),
(1,2,1,3),	(2,3,2,2),	(3,1,3,1),	(1,2,1,3),	(2,3,3,1),	(3,1,2,2),
(1,2,1,3),	(2,1,2,1),	(3,3,3,2),	(1,2,1,3),	(2,3,3,2),	(3,1,2,1),

while in $R(M)$ we have

III		
(1,2,3),	(2,3,1),	(3,1,2),
(1,2,3),	(2,1,2),	(3,3,1),
(1,2,3),	(2,3,2),	(3,1,1),
(1,2,3),	(2,1,1),	(3,3,2).

Since $\phi_1 = e_1 + e_3$, $\phi_2 = e_2$, $\phi_3 = e_4$, it is not difficult to identify the sets in group III with those in group I. Thus

$$(1, 2, 3)_{\phi_j} = \phi_1 + 2\phi_2 + 3\phi_3 = e_1 + 2e_2 + e_3 + 3e_4 = (1, 2, 1, 3)_{e_i},$$

and similarly $(2, 3, 2)_{\phi_j} = (2, 3, 2, 2)_{e_i}$, $(3, 1, 1)_{\phi_j} = (3, 1, 3, 1)_{e_i}$.

UNIVERSITY OF ALABAMA

TRIPLE SYSTEMS AS RULED QUADRICS*

W. G. WARNOCK

1. **Introduction.** If n elements $x_1, x_2, \cdots, x_n$ can be arranged in triples such that each pair x_ix_j occurs in one and only one triple, the arrangement so formed is a simple triple system. Credit for the first published paper on such systems is given to Kirkman.† Methods of construction, properties, and forms of interpretation of these and more general multiple systems can be found throughout the mathematical literature since that date.‡ In this note I propose to treat the element as a generic line in an ordinary three-space. Likewise, I shall point out some of the group properties which seem worthy of men-

* Presented to the Society, November 28, 1936.

† Kirkman, The Cambridge and Dublin Mathematical Journal, vol. 2 (1847), pp. 192–204.

‡ See A. Emch, *Triple and multiple systems, their geometric configurations and groups*, Transactions of this Society, vol. 31 (1929), pp. 25–42. An almost complete list of references is given in this paper. A more recent discussion of multiple systems is to be found in an article by R. D. Carmichael, *Tactical configurations of rank two*, American Journal of Mathematics, vol. 53 (1931), pp. 217–240.

tion. This note is restricted to simple triple systems on seven elements.

An example of a triple system on seven elements is

$$\begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 4 & 5 & 4 & 5 & 4 & 6 \\ 3 & 7 & 6 & 6 & 7 & 5 & 7 \end{matrix}. \tag{1}$$

It is well known that this system is invariant under a group G_{168}, the simple group of order 168. There are $7!/168=30$ distinct triple systems on seven elements. Cayley* showed that any 6 distinct systems contained all the 35 possible couples but that 5 systems cannot be formed to include all these couples. The 30 distinct systems are listed at the end of this article and shall be referred to by the corresponding numbers to their left.

2. **Elements interpreted as lines.** When one considers an element as a skew line in an S_3, a triple represents three skew lines. These lines may be taken as three rulings on a ruled quadric surface formed by all the lines that intersect these three rulings. A simple triple system Δ_7^3, represents seven such quadric surfaces. We examine first the nature of the intersections of these surfaces. Upon examination of (1) in §1 it is seen that the ruled quadric 123 intersects the ruled quadrics 147 and 156 in the ruling 1; 123 intersects 246 and 257 in ruling 2; likewise, 123 intersects 345 and 367 in ruling 3. Similarly the other intersections may be counted. If two ruled quadrics intersect along a ruling, the remaining portion of the intersection is a space cubic. This residual cubic cuts the common ruling in two points, as is easily shown. There are 21 residual cubics in the total system of a simple Δ_7^3. Each cubic cuts 2 rulings, and each ruling intersects 15 of the 21 cubics. On each surface lie 6 cubics. If one projects upon any plane from a point on a common ruling, the residual cubic becomes a plane cubic with a double point. The residual cubics are then rational, and the 15 which intersect a given ruling project into a system of 15 plane cubics having a common double point.

3. **Triple systems with no common triple.** There exist among the 30 distinct Δ_7^3 systems certain ones which have no triple in common with any given Δ_7^3. On seven elements these do have common couples, however. The systems (1) and (19) are two such systems. One might ask for a triple system somewhat more general than a simple system and demand that the n elements form a triple system with every

* Cayley, *Collected Papers*, vol. 1, pp. 481–484.

couple taken twice. If x represents the number of triples in such a system, it develops that $x=n(n-1)/3$. Integral solutions are possible when n is of the form $3k$ or $3k+1$. In the case under consideration, $n=7$ and $x=14$; however, for our purposes, it is better to consider this not as a system of 14 triples on 7 elements but as two simple systems without a common triple but with common couples.*

The operators

$$S_1 = (2\ 7)(3\ 5)(4\ 6), \quad S_2 = (1\ 5)(3\ 4)(6\ 7), \quad S_3 = (1\ 6)(2\ 5)(4\ 7),$$
$$S_4 = (1\ 2)(3\ 6)(5\ 7), \quad S_5 = (1\ 4)(2\ 6)(3\ 7), \quad S_6 = (1\ 3)(4\ 5)(3\ 6),$$
$$S_7 = (1\ 3)(2\ 4)(5\ 6)$$

leave the elements 1, 2, 3, 4, 5, 6, 7 of the systems (1) and (19) invariant. The product $S_1S_2=(1\ 5\ 4\ 7\ 2\ 6\ 3)$ is a cyclic substitution; so all of its powers are in the group and likewise are products S_iS_j. The group which leaves the two systems invariant is then a $G_{14}=I$, S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, T, T^2, T^3, T^4, T^5, T^6.

In terms of ruled quadrics, it is seen that each quadric intersects 9 other quadrics, 6 of its own system and 3 of its complementary system, in a common ruling and a residual cubic; 63 cubics are so formed. Again, a triple of one system has a couple in common with three triples of the complementary system. Geometrically these quadrics will have two rulings in common, leaving conics for the residual curves of intersection. This gives $7\cdot 3=21$ conics of intersection in two systems of this sort.

To find the number of distinct triple systems with no triple in common with (1), notice that the operator S_1 leaves the element 1 invariant and permutes the remaining elements in such a manner that no couple remains unchanged. If this were not demanded, a triple would remain unchanged, and thereby a $\Delta_7{}^3$ would be created with at least one triple in common with (1). With this restriction there are only 8 ways in which the element 1 can remain invariant. To these 8 permutations correspond distinct triple systems. Each system is invariant with (1) under a group of order 14 as is indicated above. These systems having common couples but no common triples with (1) are (9), (12), (15), (16), (19), (23), (26), (29) and may be obtained by operating upon (1), respectively, by

$$O_1 = (2\ 6)(3\ 7)(4\ 5), \quad P_1 = (2\ 5)(3\ 7)(4\ 6), \quad Q_1 = (2\ 7)(3\ 6)(4\ 5),$$
$$R_1 = (2\ 4)(3\ 6)(5\ 7), \quad S_1 = (2\ 7)(3\ 5)(4\ 6), \quad U_1 = (2\ 4)(3\ 5)(6\ 7),$$
$$V_1 = (2\ 5)(3\ 4)(6\ 7), \quad W_1 = (2\ 6)(3\ 4)(6\ 7).$$

* Emch, loc. cit., p. 29.

There is no group under which these 9 systems are invariant; for the product of two operators above leaves a triple invariant and hence cannot belong to the system.

4. **Systems with one triple in common.** Some of the 30 systems have 1 triple in common with (1). The system (2) furnishes an example with the triple 123 in common. The corresponding common ruled quadric has 12 residual cubics of intersections, 6 from each system, on its surface. On any other ruled quadric of the two systems lie 8 residual cubics, 6 by intersections with surfaces of its own system and 2 by intersections with surfaces of its complementary system. Sixty cubics are so formed by two systems of this type. The total number of conics formed by common couples of lines as part of the intersection is 21.

The group that leaves two such systems invariant must leave the common triple invariant. It can be expressed in terms of the remaining four elements not in this common triple. This group must be the symmetric group G_{24} or one of its subgroups. It cannot contain a transposition; for, in that case, at least one other triple would remain invariant. As this is the only restriction necessary, the desired group is the alternating group G_{12} on the elements not contained in the common triple.

There are 14 of the 30 distinct Δ_7^3 systems which have one triple in common with (1). They are (2), (3), (7), (8), (10), (13), (17), (18), (20), (21), (24), (25), (28), (30). If a couple is considered fixed, a triple can be formed in five ways by taking one of the remaining elements. This fixes the common triple and likewise fixes one element in each of the remaining triples if the process is considered as a permutation upon (1). The remaining places in the second and third triples (columns of (1)) are to be chosen from the remaining four elements, and the filling of these spaces determines the respective positions of these four elements in the last four triples. This can be done in $4\cdot 3/2=6$ ways, but the arrangement of columns in a system is nonessential so that finally there are $5\cdot 6/2=15$ possible arrangements. These include (1); the other Δ_7^3 systems have been listed above.

5. **Systems with three triples in common.** The 7 remaining Δ_7^3, systems have three triples in common with (1). Each of these is invariant with (1) under a group of order 2^3 and type (1, 1, 1).

From the fact that there are $7\cdot 6/2=21$ ways of forming a transposition substitution and the fact that the invariant group is of the type (1, 1, 1), it follows that there are $21/3=7$ distinct Δ_7^3 systems of this nature. These are (4), (5), (6), (11), (14), (22), (27).

In terms of the ruled quadrics and their intersections, there lie 10 residual cubics on each of the common quadrics of the two complementary systems and 6 on each of the other quadric surfaces. On each of these surfaces lie 3 conics. This gives 36 cubics and 12 conics as curves of intersection of (1) with each of the seven systems above. This completes the count of the simple systems.

List of distinct Δ_7^3 systems

	1	1	1	2	2	3	3
(1)	2	5	4	5	4	4	6
	3	6	7	7	6	5	7
	1	1	1	2	2	3	3
(2)	2	4	5	4	6	5	4
	3	6	7	5	7	6	7
	1	1	1	2	2	3	3
(3)	2	4	6	4	5	5	4
	3	5	7	7	6	7	6
	1	1	1	2	2	3	3
(4)	2	4	3	4	6	4	5
	3	7	6	5	7	6	7
	1	1	1	2	2	3	3
(5)	2	4	6	4	5	4	5
	3	5	7	6	7	7	6
	1	1	1	2	2	3	3
(6)	2	4	5	5	4	4	6
	3	6	7	6	7	5	7
	1	1	1	2	2	3	4
(7)	2	3	5	5	3	4	6
	4	6	7	6	7	5	7
	1	1	1	2	2	3	4
(8)	2	3	6	3	5	4	5
	4	5	7	6	7	7	6
	1	1	1	2	2	3	4
(9)	2	3	6	5	3	4	5
	4	5	7	6	7	6	7
	1	1	1	2	2	3	4
(10)	2	3	6	3	7	4	5
	4	7	5	5	6	6	7
	1	1	1	2	2	3	4
(11)	2	3	5	3	5	4	6
	4	7	6	6	7	5	7
	1	1	1	2	2	3	4
(12)	2	3	5	3	6	4	5
	4	6	7	5	7	7	6
	1	1	1	2	2	3	4
(13)	2	3	6	3	4	5	5
	5	4	7	7	6	6	7
	1	1	1	2	2	3	5
(14)	2	3	4	4	3	4	6
	5	6	7	6	7	5	7
	1	1	1	2	2	3	4
(15)	2	3	6	4	3	5	5
	5	4	7	7	6	7	6
	1	1	1	2	2	3	4
(16)	2	3	4	3	6	5	5
	5	7	6	4	7	6	7
	1	1	1	2	2	3	4
(17)	2	3	4	3	6	5	5
	5	6	7	4	7	7	6
	1	1	1	2	2	3	5
(18)	2	3	4	3	4	4	6
	5	7	6	6	7	5	7
	1	1	1	2	2	3	4
(19)	2	3	5	3	4	5	6
	6	4	7	7	5	6	7
	1	1	1	2	2	3	5
(20)	2	3	4	3	4	4	6
	6	5	7	7	5	6	7
	1	1	1	2	2	3	4
(21)	2	3	4	5	3	5	6
	6	7	5	7	4	6	7
	1	1	1	2	2	3	4
(22)	2	3	4	3	5	6	5
	6	5	7	4	7	7	6
	1	1	1	2	2	3	5
(23)	2	3	4	4	3	4	6
	6	7	5	7	5	6	7
	1	1	1	2	2	3	4
(24)	2	3	5	3	4	6	5
	6	4	7	5	7	7	6
	1	1	1	2	2	3	5
(25)	2	3	4	5	3	6	5
	7	5	6	6	4	7	7
	1	1	1	2	2	3	4
(26)	2	4	3	3	5	5	6
	7	5	6	4	6	7	7
	1	1	1	2	2	6	7
(27)	2	3	5	3	4	6	5
	7	4	6	5	6	7	7
	1	1	1	2	2	3	5
(28)	2	3	4	3	4	4	6
	7	6	5	5	6	7	7
	1	1	1	2	2	3	5
(29)	2	3	4	3	4	4	6
	7	5	6	6	5	7	7
	1	1	1	2	2	3	4
(30)	2	3	5	3	4	5	6
	7	4	6	6	5	7	7

University of Alabama