GROUPS OF MOTIONS IN CONFORMALLY FLAT SPACES. II

JACK LEVINE

1. Introduction. In a previous paper with a similar title,* we have shown that all groups of motions admitted by a conformally flat metric space $V_n$ must be subgroups of the general conformal group $G_N$ of $N={1/3}(n+1)(n+2)$ parameters generated by

$$ (1) \quad \xi^i = b^i + a_0 x^i + x^i a_j x^j - \frac{1}{2} a_s e_i e_j (x^j)^2 + b_i x^i, \quad e_i = \pm 1. $$

In (1), the $b^i$ satisfy the relations $e_i b_j^i + e_j b_i^j = 0$, $(i, j$ not summed). Otherwise the $a$'s and $b$'s in (1) are arbitrary.

To define a group of motions of $V_n$, the $\xi^i$ must satisfy the equations

$$ (2) \quad \xi^k \frac{\partial h}{\partial x^k} + h \frac{\partial \xi^i}{\partial x^i} = 0, \quad i \text{ not summed}, $$

and the coordinates $x^i$ of (2) are such that $g_{ij} = e_i \delta_j^i h^2$. Hence in this coordinate system, the metric has the form

$$ (3) \quad ds^2 = h^2 \sum e_i (dx^i)^2. $$

In this paper we shall consider the simplest subgroups of $G_N$, and determine the nature of the function $h$ corresponding to each. Also we give a restatement of Theorem 2 of I, since it is not complete as given.

2. The group $G_N$. The basis of the group $G_N$ may be taken in the form

$$ (4) \quad P_i = p_i, $$

$$ (5) \quad S_{ij} = e_i x^j p_i - e_j x^i p_i, \quad i, j \text{ not summed}, $$

$$ (6) \quad U = x^i p_i, $$

$$ (7) \quad V_i = 2 x^i x^j p_j - e_i e_j (x^j)^2 p_i, $$

where $p_i = \partial / \partial x^i$; and its commutators are†

---

*Groups of motions in conformally flat spaces,* this Bulletin, vol. 42 (1936), pp. 418–422. The results of this paper (which we refer to as I) will be assumed known.

† All small Latin indices take the values $1, 2, \cdots, n$, with $n > 2$, unless otherwise noted.

MOTIONS IN FLAT SPACES

(8a) \((P_i, P_j) = 0,\)

(8b) \((P_i, U) = P_i,\)

(8c) \((P_i, S_{jk}) = \varepsilon_{ik}P_k - \varepsilon_{jk}P_i,\)

(8d) \((P_i, V_j) = 2\delta_{ij}U - 2\varepsilon_{ij}S_{ij},\)

(8e) \((S_{ij}, S_{kl}) = \varepsilon_{jl}S_{ik} - \varepsilon_{jk}S_{il} - \varepsilon_{ik}S_{jl} + \varepsilon_{il}S_{jk},\)

(8f) \((S_{ij}, U) = 0,\)

(8g) \((S_{ij}, V_k) = \varepsilon_{ik}V_j - \varepsilon_{jk}V_i,\)

(8h) \((U, V_i) = V_i,\)

(8i) \((V_i, V_j) = 0.\)

The four types of symbols, \(P_i, S_{ij}, U, V_i,\) will be considered singly and in various combinations to form the subgroups to be discussed.

3. Subgroups of one type of symbol. We consider first the subgroups with symbols* 

(a) \([P_a],\)  
(b) \([U],\)  
(c) \([S_{ab}],\)  
(d) \([V_a].\)

The notation \([P_a]\) means \([P_1, P_2, \cdots, P_r]\), and similarly for other expressions of this nature. That each of (a)–(d) forms a subgroup follows from (8a), (8e), (8i).

For (a), we have from (4), \(\xi^i_a = \delta^i_a,\) and (2), written in the form

\[ \xi_k^a \frac{\partial h}{\partial x^k} + h \frac{\partial \xi^i_a}{\partial x^i} = 0, \]

becomes

\[ \frac{\partial h}{\partial x^a} = 0. \]  

(9)

Hence (a): \(h = h(x^{r+1}, \cdots, x^n).\) In case \(r = n,\) \(h\) is constant, and the \(V_n\) is flat.

The finite equations of the group \([P_a]\) are

\[ x'^i = x^i + \alpha^i_a \delta^j_a \]

(10)

with parameters \(\alpha^i_a.\) Because of the form of (10), we call this group the \(T_r\) of translations. However, the group of motions \([P_a]\) is not a group of translations of the \(V_n\) unless \(h = \text{constant},\) † that is, unless \(V_n\) is flat.

* Greek letters take the values 1, 2, \(\cdots, r,\) with \(r \leq n.\)

† L. P. Eisenhart, Continuous Groups of Transformations, p. 212. We refer to this book as CG.
For (b), we have $\xi^i = x^i$, and (2) becomes

$$
(11) \quad x^i \frac{\partial h}{\partial x^i} = -h.
$$

Hence $h$ is homogeneous of degree $-1$, that is,

$$
(12) \quad h = \frac{1}{x^1} \phi \left( \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right),
$$
say, where $\phi$ is an arbitrary function of its arguments.

The finite equations of the group $[U]$ are $x'^i = ax^i$, the group of dilations.

For (c), we find

$$
\xi_{\alpha \beta} = e_\alpha \delta_\beta^i x^\alpha - e_\beta \delta_\alpha^i x^\beta,
$$
as the vector components of the group $[S_{\alpha \beta}]$ of $\frac{1}{2}r(r-1)$ parameters. The equations (2) which must be satisfied for each $\xi_{\alpha \beta}$ now become

$$
(12) \quad X_{\alpha \beta} h = e_\alpha x^\alpha \frac{\partial h}{\partial x^\beta} - e_\beta x^\beta \frac{\partial h}{\partial x^\alpha} = 0, \quad \alpha, \beta \text{ not summed}.
$$

These equations have as general solution,

$$
(13) \quad h = h(u; x^{r+1}, \ldots, x^n),
$$

where $u = \sum e_\alpha (x^\alpha)^2$.

In obtaining this, we use the fact that the system (12) contains $r-1$ independent equations, since

$$
e_\alpha x^\alpha X_{\beta \gamma} + e_\beta x^\beta X_{\gamma \alpha} + e_\gamma x^\gamma X_{\alpha \beta} = 0,
$$

no summing,

and it is also a complete system.*

The group $[S_{\alpha \beta}]$ has the finite equations

$$
x'^\alpha = a_\alpha x^\beta, \quad x'^A = x^A, \quad A = r + 1, \ldots, n,
$$

with

$$
\sum e_\alpha a_\beta a_\gamma = e_\beta \delta_\gamma^\alpha.
$$

We call this group of $\frac{1}{2}r(r-1)$ parameters, the $R_{r(r-1)/2}$ of rotations.†

The vector components for the group (d) are

$$
\xi^i = 2x^i x^\alpha - e_\alpha \delta_\gamma^i R,
$$

† CG, p. 57, problem 12.
where \( R = \sum e_i (x^i)^2 \). Equations (2) reduce, for this case, to

\[
2x^a x^i \frac{\partial h}{\partial x^i} - e_a R \frac{\partial h}{\partial x^a} + 2hx^a = 0.
\]

If we put \( \lambda = x^i \partial h / \partial x^i \), (13) may be written in the form

\[
\frac{2(\lambda + h)}{R} = \frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha}, \quad \alpha \text{ not summed.}
\]

Since the left member of this equation is independent of \( \alpha \), we may write

\[
\frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha} = \frac{e_\beta}{x^\beta} \frac{\partial h}{\partial x^\beta},
\]

which simplifies to (12), and hence \( h \) is of the form for (c). Using this form for \( h \) in (13), we obtain on reduction,

\[
(u - v) \frac{\partial h}{\partial u} + \sum x^A \frac{\partial h}{\partial x^A} = -h, \quad A = r + 1, \ldots, n,
\]

with

\[
v = \sum e_A (x^A)^2.
\]

The equation (14) has as solution

\[
h = \frac{1}{R} \phi \left( \frac{x^{r+1}}{R}, \ldots, \frac{x^n}{R} \right).
\]

In case \( r = n \), \( h = a/R \), with a constant, and the \( V_n \) is flat.*

The finite equations for the group \([V_a]\) are†

\[
x^i = \frac{x^i - \frac{1}{2} R \delta_a e_a a_a}{1 - a_a x^a + \frac{1}{4} e_a e_\beta a_\alpha^2 (x^\beta)^2}.
\]

4. Subgroups with two types of symbols. We consider in this section the simplest subgroups with two types of symbols. These are:

(e) \([P_\alpha, S_\beta]\),

(f) \([P_\alpha, U]\),

(g) \([S_\alpha, U]\),

(h) \([V_\alpha, U]\),

(i) \([S_\alpha, V_\gamma]\).

Each of these we discuss briefly.

(e). The function \( h \) has the same form as for (a) since equations (12) are satisfied identically if (9) are.


† Lie, loc. cit., p. 350.
(f). Using the form of \( h \) for (a) in (11), we see that \( h \) is homogeneous of degree \(-1\) in \( x^{r+1}, \ldots, x^n \), that is, we may write

\[
h = \frac{1}{x^{r+1}} \phi \left( \frac{x^{r+2}}{x^{r+1}}, \ldots, \frac{x^n}{x^{r+1}} \right).
\]

If \( r = n \), there is no solution.

(g). If we substitute for \( h \) in (11) its value as determined from (c), we obtain

\[
2u \frac{\partial h}{\partial u} + x^A \frac{\partial h}{\partial x^A} = -h, \quad A = r + 1, \ldots, n.
\]

Hence,

\[
h = \frac{1}{u^{1/2}} \phi \left( \frac{x^{r+1}}{u^{1/2}}, \ldots, \frac{x^n}{u^{1/2}} \right).
\]

(h). Equations (11) and (13) show \( \partial h/\partial x^a = 0 \), so that \( h \) is the same as in (f). If \( r = n \), there is no solution.

(i). For (d), we have seen that (13) imply (11), that is, the form of \( h \) for (i) is the same as that for (d).

5. **Subgroups with three and four types of symbols.** Of the four possibilities \([P_a, S_{\beta \gamma}, V_\delta], [P_a, S_{\beta \gamma}, U], [P_a, V_\beta, U], [S_{\alpha \beta}, V_\gamma, U]\), only the second and fourth give subgroups:

(j) \([P_a, S_{\beta \gamma}, U]\),

(k) \([S_{\alpha \beta}, V_\gamma, U]\).

For (j), the \( P_a, S_{\beta \gamma} \) imply \( h = h(x^{r+1}, \ldots, x^n) \), and then \( U \) shows \( h \) is the same form as in (f). There is no solution of \( r = n \).

The form of \( h \) for (k) will be the same for (h), as follows from (i), that is, \( h \) will have the same form as for (f). If \( r = n \), there is no solution.

The simplest four type symbol subgroup is

(l) \([P_a, V_\beta, S_{\gamma \delta}, U]\).

It is easily seen that the solution for \( h \) is the same as for (f), and there is no solution for \( r = n \).

6. **Indices in different ranges.** So far, we have considered only subgroups whose symbol indices all have the same range, \( 1, \ldots, r \). In this section we discuss cases (e), (i), (j), (k), and (l) with the indices for the various types of symbols in different ranges.

Case (m): \([P_i, S_{\beta j}]\). Let \( i \) range through \( 1, \ldots, r \), and \( j, k \) through any set of \( t \) indices, \( s_1, s_2, \ldots, s_t \), with \( s_1 < s_2 < \cdots < s_t \). Then either:
For case (m₁), equations (9) imply (12) with α, β in the range $s₁, s₂, \ldots, sₙ$. Hence $h$ has the same form as in (a).

In the second case, (m₂), there must be a common index in $(1, \ldots, r)$ and $(s₁, \ldots, sₙ)$, say $β$. Then, in (8c), choose $i=j=β$, and $k=sₙ$. This gives

$$(P_β, S_{βₙ'}) = e_β P_{s'ₙ},$$

which is not in the set $[P_α]$. Hence, this case is impossible.

For case (m₃), the two sets of indices have no index in common, and we must have $t≥2$. Without loss of generality, we may take the set $s₁, \ldots, sₙ$ to be $r+1, r+2, \ldots, r+t$. The form of $h$ is easily seen to be

$$h = h(vₜ; x^{r+t+1}, \ldots, xⁿ), \quad vₜ = \sum_{r+t} e_j(x^j)^2.$$  

Case (n): $[S_{jk}, Vₜ]$. As in case (m), there are three possibilities, only the first and third being possible. If we let $i$ take the range $1, \ldots, r$, then if $sₙ≤r$, $h$ has the same form as for (d). If $sₙ>r$, we may let $j, k$ have the range $r+1, r+2, \ldots, r+t$. Then $h$ must satisfy (13), and (12) with the indices in this latter range. Since (13) implies (12), we must have $h = h(u; vₜ; x^{r+t+1}, \ldots, xⁿ)$. Using this form for $h$ in (13), we obtain

$$(u - w) \frac{∂h}{∂u} + vₜ \frac{∂h}{∂vₜ} + x^B \frac{∂h}{∂x^B} = -h, \quad B = r+t+1, \ldots, n,$$

with $w = \sum e_B(x^B)^2$. This equation has as solution

$$h = \frac{1}{R - vₜ} \phi \left( \frac{vₜ}{R - vₜ}; \frac{x^{r+t+1}}{R - vₜ}, \ldots, \frac{xⁿ}{R - vₜ} \right).$$

With three types of symbols, we consider first $[P_α, S_{jk}, U]$, and let $i=1, \ldots, r$. If the indices of $S_{jk}$ are all contained in the range $1, \ldots, r$, $h$ has the same form as for $[P_α, U]$. Otherwise, we must have all $j, k$ indices outside the range $1, \ldots, r$. Then we have:

(o) $[P_α, S_{jk}, U]$, and $h = h(vₜ; x^B)$, using the notation of case (n). With this value of $h$ in (11) we obtain equation (15) with $u$ replaced by $vₜ$. Hence,

$$h = \frac{1}{v₁^{1/2}} \phi \left( \frac{x^{r+t+1}}{v₁^{1/2}}, \ldots, \frac{xⁿ}{v₁^{1/2}} \right).$$

As the next case we consider $[V_α, S_{jk}, U]$. If the $j, k$ indices are included in $1, \ldots, r$, we get the same form for $h$ as in $[V_α, U]$. If not
we must have \( j, k \) in the range \( J, K \), to give: (p) \([V_a, S_{JK}, U]\). The symbols \( V_a, U \) imply \( h = h(x^{r+1}, \ldots, x^n) \), and then the symbols \( S_{JK} \) imply \( h = h(v_t; x^B) \), the same as in (o).

The other two possibilities \([P_t, S_{jk}, V_t]\), \([P_t, V_t, U]\) are easily shown to be impossible, no matter in what ranges we choose the indices of the various symbols.

For four types we have \([P_a, S_{jk}, V_t, U]\). If \( j, k \) are in the \( J, K \) range, we have a contradiction from \((P_a, V_t)\), no matter what range \( l \) has. The only other choice is \( j, k \) included in the \( 1, \ldots, r \) range. Then, from \((P_a, V_t)\), we must have \( l \) in this range also. This gives

\[
\begin{align*}
\text{(q) } [V_a, S_{\alpha'\beta'}, V_{\gamma'}, U], & \quad \alpha', \beta', \gamma' \text{ range included in } 1, \ldots, r, \\
\text{and } h \text{ has the same form as for (f), as easily follows.}
\end{align*}
\]

7. Summary. We give here a summary of the various forms for \( h \) corresponding to the subgroups considered.

(a) \([P_a]\), \( h = h(x^{r+1}, \ldots, x^n) \);

(b) \([U]\), \( h = \frac{1}{x^1} \phi \left( \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right) \);

(c) \([S_{\alpha\beta}]\), \( h = h(u; x^{r+1}, \ldots, x^n) \);

(d) \([V_a]\), \( h = \frac{1}{R} \phi \left( \frac{x^{r+1}}{R}, \ldots, \frac{x^n}{R} \right) \);

(e) \([P_a, U]\), \( h = \frac{1}{x^{r+1}} \phi \left( \frac{x^{r+2}}{x^{r+1}}, \ldots, \frac{x^n}{x^{r+1}} \right), r = n, \text{no solution;} \)

(f) \([S_{\alpha\beta}, U]\), \( h = \frac{1}{u^{1/2}} \phi \left( \frac{x^{r+1}}{u^{1/2}}, \ldots, \frac{x^n}{u^{1/2}} \right) \);

(g) \([P_a, S_{IJ}]\), \( h = h(u; x^B) \);

(h) \([V_a, S_{IJ}]\), \( h = \frac{1}{R - v_t} \phi \left( \frac{v_t}{R - v_t}; \frac{x^B}{R - v_t} \right) \);

(i) \([P_a, S_{\beta'\gamma}]\), \( h \) as in (a);

(j) \([S_{\alpha\beta}, V_{\gamma}]\), \( h \) as in (d);

(k) \([P_a, S_{\beta'\gamma}, U]\), \( h \) as in (o);

(l) \([P_a, V_{\beta}, S_{\gamma'}, U]\), \( h \) as in (p);

(m) \([P_a, S_{\beta'\gamma}, U]\), \( h \) as in (q).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
all have $h$ as in (f);

\[(p_s) \quad [V_\alpha, S_{ij}, U], \quad h \text{ as in } (o_s)\].

In the above summary we have used the following notation:

$$R = \sum e_i(x^i)^2, \quad u = \sum e_\alpha(x^\alpha)^2, \quad v_i = \sum e_\ell(x^\ell)^2,$$

$i = 1, \ldots, n$; Greek letters have the range $1, \ldots, \gamma; I, J = \gamma + 1, \ldots, \gamma + \kappa; A = \gamma + 1, \ldots, n$; primed Greek letters have a range contained within $1, \ldots, \gamma; B = \gamma + \kappa + 1, \ldots, n$.

8. Restatement of Theorem 2 of I. In the proof of this theorem, the possibility $a_0 = b^i = a_i = 0$ was omitted. In this case, $\xi^i$ has the form $\xi^i = b^i x^i$, and the function $f(R)$ is arbitrary. The group for this case evidently the rotation group $[S_{ij}]$ of $\frac{1}{2}n(n-1)$ parameters. It is not difficult to show that the subgroups corresponding to the two cases mentioned in the theorem are $[ce_\iota P_i + V_\iota, S_{jk}]$ for $f(R) = (\alpha R + \beta)^2$ and $[S_{ij}, U]$ for $f(R) = \alpha R$. We may thus state the corrected theorem in the form:

**Theorem.** Every metric space with quadratic form $\sum e_i(dx^i)^2/f(R)$ admits the rotation group $[S_{ij}]$ as a group of motions. The only metric spaces with this quadratic form which admit other groups of motions are spaces of constant curvature, and $f$ has the form $f(R) = (\alpha R + \beta)^2$, and the group is $[ce_\iota P_i + V_\iota, S_{jk}]$, and spaces with $f(R) = \alpha R$, in which case the group is $[S_{ij}, U]$.

NORTH CAROLINA STATE COLLEGE