From (7.15), (7.17), (7.18), we conclude that these $\infty^1 V_{m-1}$ contain axial points. Hence the theorem:

**Theorem.** The $V_m$ in $S_n$ of type (3) are either $V_m$ in $S_{m+2}$ or contain $\infty^1 V_{m-1}$ with axial points.

If the rank of $h^\mu$ (index $n-1$) is greater than two, then the rank of $\mathcal{E}_{\alpha}h^\alpha$ ($h$ of index $n-1$) is greater than one. In this case, from Segre’s theorem, the $\infty^1 V_{m-1}$ lie in $\infty^1 S_m$. If the rank of $h^\mu$ (index $n-1$) is two and its nonzero domain* does not contain the nonzero domain of $h^\mu$ (index $n$), then the same result is valid; if it does contain the nonzero domain of this $h^\mu$, then $\mathcal{E}_{\alpha}h^\alpha$ (index $n-1$) is of rank one. In this last case the $\infty^1 V_{m-1}$ are $\infty^1$ developable $S_{m-1}$.

From §§6 and 7, we have the theorem:

**Theorem.** If the rank of any of the two second fundamental tensors is greater than two, then $V_m$ in $S_n$ with planar points are (1) $V_m$ consisting of $\infty^1 V_{m-1}$ imbedded in $\infty^1 S_{m+1}$, or (2) $V_m$ consisting of $\infty^1 V_{m-1}$ imbedded in $\infty^1 S_m$, or (3) $V_m$ lying in $S_{m+2}$.

---

**ON TRANSITIVE GROUPS THAT CONTAIN CERTAIN TRANSITIVE SUBGROUPS**

**W. A. MANNING**

If a simply transitive permutation group $G$ of compound degree $n$ contains a regular abelian subgroup $H$ of order $n$, and if at least one Sylow subgroup of $H$ is cyclic, $G$ is imprimitive. The proof of this important theorem, due to Wielandt,‡ is remarkable for its brevity. But familiarity with certain preliminary theorems of Schur’s§ is assumed. Unfortunately these theorems, as presented by Schur, do not appear to be as elementary as they really are. It seems, therefore, worth while to offer a complete proof of Wielandt’s theorem that is elementary throughout, free from the theories of rings and representations, and based on the fundamental concept of the double coset, introduced by Cauchy|| in 1846. Some generalizations, too, can readily be made.

* Vol. 1, p. 19; German “Gebiet.”
† Presented to the Society, December 29, 1938, under the title *A note on transitive groups with regular subgroups of the same degree.*
§ I. Schur, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, p. 598.
In §4 of this paper it will be shown how, when a transitive group is given, the set of all linear homogeneous substitutions on the same \( n \) variables which are commutative with every permutation of the group can be easily and directly obtained.

1. **Double cosets and primary subsets.** The \( n \) letters permuted by the transitive group \( G \) of order \( ng \) are \( a, b_1, \cdots \), and \( G_a \) is the subgroup of \( G \) that fixes the letter \( a \). \( G_a \) is to be regarded as an intransitive group of degree \( n \) with \( r \) transitive constituents, counting those on one letter. If \( ng > n \), and \( r = 2 \), \( G \) is doubly transitive.

If \( s \) is a permutation of \( G \), the double coset \( G_a s G_a \) is a collection* of \( g^2 \) elements, each occurring \( g/m \) times if \( m \) is the number of letters in the transitive constituent of \( G_a \) to which the letter \( x \) belongs in case \( s \) replaces \( a \) by \( x \). If \( s \) is the identical permutation \( e \), \( G_a e G_a = G_a G_a \), which can be written \( G^2_a \) or \( gG_a \), a collection in which each element of the set \( G_a \) is repeated \( g \) times. The inverse double coset \( (G_a s G_a)^{-1} = G_a s^{-1} G_a \) has the same number \( mg \) of distinct permutations as \( G_a s G_a \), so that, if \( s = (yax \cdots) \cdots, y \) is one of the letters of a transitive constituent of \( G_a \), also of degree \( m \). This is the "pairing" of transitive constituents discovered by Burnside.† The double coset \( G_a s G_a \) is unchanged if \( s \) is replaced by any other permutation of \( G_a s G_a \).

Let the crosscut of two finite collections \( K \) and \( L \) be indicated by the symbol \( K \cap L \), and be defined by saying that if an element is repeated \( k \) times in \( K \) and \( l \) times in \( L \), it occurs \((k + l - |k - l|)/2 \) times in \( K \cap L \).

Let \( H \) be a transitive subgroup of degree \( n \) and order \( nh \) of the transitive group \( G \). The subgroup of \( H \) that fixes \( a \) is \( H_a \), and is of order \( h \). The crosscut \( H \cap (G_a s G_a) \) will, if \( s \) is a permutation of \( G \), be called a primary subset of \( H \) with respect to \( G_a \). The subset \( H_a \) of \( H \) is a primary subset of \( H \) with respect to \( G_a \), even when \( h = 1 \). If \( s = s_1 = (ab_1 \cdots) \cdots \), where \( b_1 \) is one of the \( m \) letters \( b_1, b_2, \cdots, b_m \), of a transitive constituent of \( G_a \), this crosscut consists of \( mh \) distinct permutations of \( H \):

\[
K: \quad s_1 = (ab_1 \cdots) \cdots, s_2 = (ab_2 \cdots) \cdots, \cdots, \\
s_m = (ab_m \cdots) \cdots; \cdots.
\]

If a collection consists of these \( mh \) permutations of a primary subset,

---

each repeated $n_1$ times, it is written $n_1K$. A chief collection of $H$ with respect to $G_a$ is a collection of primary subsets of $H$ with respect to $G_a$, and if $K_1, K_2, \ldots$ are the primary subsets involved in the chief collection $K$, we may use the notation

$$K = n_1K_1 + n_2K_2 + \cdots + n_qK_q,$$

where $n_1, n_2, \ldots$ are natural numbers. A primary subset as here defined is, when $H$ is regular, equivalent to the "primären Komplex" of Schur, Wielandt and Kochendörffer, but as there is confusion in the use of the word "Hauptkomplex," I prefer to avoid the word "complex" altogether.

It results immediately from the definition that $H$ has exactly $r$ primary subsets, with respect to $G_a$ corresponding to the $r$ transitive constituents of $G_a$, and any given permutation of $H$ is in one and only one primary subset (with respect to $G_a$). If $K$ is a primary subset of $H$ with respect to $G_a$, it is a primary subset of $H$ with respect to every subgroup $G_z$ that fixes one letter $x$ of $G$ if $K$ is invariant under $H$.*

In particular, the phrase "with respect to $G_a$" can be omitted when $H$ is abelian. Two collections $K$ and $L$ of permutations of $G$ are said to be permutable if $KL = LK$, exact account being taken of the number of times a permutation recurs. With this convention as to the product of two collections we can state a well known theorem as follows:

Two permutation groups $U$ and $V$, of orders $u$ and $v$ respectively, with $w$ permutations in common, generate a group of order $uv/w$ if and only if $UV = VU$. Then $H$ and $G_a$ are permutable groups.

2. Criteria for primitivity. Throughout this section we shall retain the preceding notation: $H$ is any transitive subgroup of degree $n$ and

* The necessary and sufficient condition that a primary subset $K$ of $H$ with respect to $G_a$ be a primary subset of $H$ with respect to every subgroup $G_b$ is that all the transforms of $K$ under $H$ are also primary subsets of $H$ with respect to $G_b$.

This theorem can be proved as follows. Let $K = H \cap G_aAG_a$ for every letter $x$ of $G$. Let $t = (xa \cdots)$ be a permutation of $H$. Then $t^{-1}Kt = H \cap Gat^-1AG_a$, because $t^{-1}Gat^-1 = G_a$. Note that for $(xa \cdots)$ we can use any one of the permutations $tH_a$, because the permutations of $H_a$ transform every primary subset of $H$ with respect to $G_a$ into itself, and $t^{-1}Kt$ is seen to be a primary subset of $H$ with respect to $G_a$ by definition. Now if $K = H \cap G_aAG_a$ for every $x$, the permutations of a certain number of cosets $tH_a, \cdots$ transform $K$ into $K$, thus forming a subgroup $N$ of $H$, and the permutations $tN, tN, \cdots$ transform $K$ into the primary subsets $K, K_1, K_2, \cdots$ with respect to $G_a$. Conversely, if all the transforms of $K$ under $H$ are primary subsets of $H$ with respect to $G_a$ we take $\pi$ permutations $t = (yax \cdots)$ from $H$, $y$ running through the $\pi$ letters. Now $t^{-1}Kt = H \cap Gat^-1AG_a = H \cap Gat^-1AG_a$, the last because $t^{-1}Kt$ is a primary subset of $H$ with respect to $G_a$ by hypothesis. Hence $K = t(t^{-1}Kt)t^{-1} = H \cap G_yAG_y$, for every letter $y$ of $G$. June 20, 1939.
order \( nh \) of a transitive group \( G \) of degree \( n \) and order \( ng \): \( H_a \) and \( G_a \) are the subgroups fixing one letter \( a \) of \( H \) and \( G \) respectively. Our fundamental theorem is the following:

**Theorem 1.** A collection \( K \) of permutations of \( H \) is a chief collection of \( H \) with respect to \( G \) if and only if \( H_aK = hK \) and \( G_aK = KG_a \).

First, let \( K \) be a primary subset of \( H \) with respect to \( G \) composed of the \( mh \) permutations of \( H_a s_1, H_a s_2, \ldots, H_a s_m \), where
\[
{s_1 = (c_1a_1 \cdots) \ldots, s_2 = (a_2 \cdots) \ldots, \ldots, s_m = (a_m \cdots) \ldots.}
\]

Evidently \( H_a K = hK \). The set of \( mg \) permutations in the \( m \) cosets \( G_a s_1, G_a s_2, \ldots, G_a s_m \) consists of the distinct permutations of the double coset \( G_a s_1 G_a \); therefore, exactly the same permutations occur in \( s_1 G_a, s_2 G_a, \ldots, s_m G_a \). Thus \( G_a K = KG_a \), when \( K \) is a primary subset. If \( K = H_a, H_a K = hK \) and \( G_a K = KG_a \). It follows that, if \( K \) is a chief collection of \( H \) with respect to \( G \), \( H_a K = hK \) and the collections \( G_a \) and \( K \) are permutable.

To prove the converse, let \( K \) be a collection of permutations of \( H \) which satisfies the two conditions of the theorem. Then \( G_a G_a K = G_a KG_a \), and we expand this by putting for \( K \) each of its permutations (call them \( s_1, s_2, \ldots \)), each as often as it occurs in \( K \), so that
\[
g G_a K = n_1(G_a s_1 G_a) + n_2(G_a s_2 G_a) + \cdots.
\]
The only permutations of \( H \) in \( G_a K \) are the permutations of \( H_a K \), which by hypothesis is \( hK \); therefore
\[
g h K = n_1(H \cap G_a s_1 G_a) + n_2(H \cap G_a s_2 G_a) + \cdots,
\]
that is, \( ghK \) is a chief collection. Therefore \( K \) is a chief collection with respect to \( G \). For, if \( K \) is not a chief collection, \( ghK \) is not one.

**Theorem 2.** If \( K \) and \( L \) are two chief collections of \( H \) with respect to \( G \), \( KL \) is a chief collection with respect to \( G \).

For \( HKL = hKL \) and \( G_a KL = KG_a L = KLG_a \).

A chief collection of \( H \) is said to be singular if its permutations generate a proper subgroup of \( H \) of order greater than \( h \).

**Theorem 3.** The group \( G \) is imprimitive if and only if some chief collection of \( H \) is singular.

Let \( G \) be imprimitive. Then \( G_a \) is a proper subgroup of a proper subgroup \( J \) of \( G \). This \( J \) does not include \( H \), but it does contain at
least one permutation of the set $H - H_a$. Let $s$ be a permutation of $J - G_a$. Since every element of $G_a s G_a$ is in $J$, $H \cap G_a s G_a$ is a subset of $H \cap J$, which last is a proper subgroup of $H$ different from $H_a$; hence $H \cap G_a s G_a$ is a singular primary subset.

Let $K$ be a singular chief collection of $H$ with respect to $G_a$. Its elements generate a proper subgroup $C$ of $H$ of order greater than $h$. Since $C$ is of finite order, there exists a natural number $N$ such that every permutation of $C$ is in the chief collection $K + K^2 + \cdots + K^N$; of course every element of this collection is in $C$, so that $C$ is a chief collection of $H$ with respect to $G_a$. By Theorem 1, $H_a$ is a subgroup of $C$, and $CG_a = G_a C$, that is, $C$ and $G_a$ are permutable groups with $H_a$ in common, and therefore $CG_a$ is a proper subgroup of $G$, which means that $G$ is imprimitive.

The following theorem is an extension of Wielandt's lemma:

**Theorem 4.** Let $P$ be a proper subgroup of $H$ but not a subgroup of $H_a$. If the elements of a set of left (or right) cosets of $H$ with respect to $P$ constitute a chief collection $K$ of $H$ with respect to $G_a$, and if $K \neq H$, $G$ is imprimitive.

If $P$ alone, or if $P$ and certain cosets with respect to $P$, constitute a chief collection $L$ of $H$ ($L \neq H$), then $K = H - L$ is a chief collection composed of cosets with respect to $P$. Hence it is sufficient to discuss the case in which $K$ does not contain $P$. Then there are at least two primary subsets involving permutations of $P$ ($e$ is in one of them) and at least one in $K$, so that $H$ has at least three primary subsets with respect to $G_a$, which means that $G_a$ has at least three transitive constituents. Let $H_a s_1, H_a s_2, \ldots, H_a s_k$, where $s_1 = (a y_1 \cdots) \cdots$, $s_2 = (a y_2 \cdots) \cdots$, $s_k = (a y_k \cdots) \cdots$, be the permutations of the collection $K$. Now $H_a s_1 P, H_a s_2 P, \ldots, H_a s_k P$ include (with repetitions) all the left cosets that make up $K$. Let $t = (y z \cdots) \cdots$ be a permutation of $P$. Since the product $s_i t = (a z \cdots) \cdots$ belongs to $K$, $z$ is one of the letters $y_1, y_2, \cdots, y_k$, and therefore $P$ permutes the letters of $P$ only among themselves. But $G_a$ also permutes these letters $y_1, y_2, \cdots, y_k$ among themselves and in consequence the group generated by $P$ and $G_a$ is intransitive. Hence $G_a$ is a proper subgroup of a proper subgroup of $G$, and $G$ is imprimitive.

The following theorem is due to Schur, and the proof which we shall briefly indicate is his.*

**Theorem 5.** Let $G$ be a primitive group and let the transitive group $H$ be abelian. If $p$ is a prime divisor of the degree $n$, and if $r$ is any permuta-

---

* I. Schur, loc. cit.
tion of $H$ other than the identity, the number of solutions of $x^p = r$ in a primary subset $K$ of $H$ is congruent to zero, modulo $p$.

Let $s_1, s_2, \ldots, s_m$ be the permutations of the primary subset $K$. In the chief collection $K^p$ the number of times any element, except perhaps $s_1^p, s_2^p, \ldots, s_m^p$, occurs is, because $H$ is abelian, a multiple of $p$. If from $K^p$ we remove those primary subsets which occur $kp$ times ($k = 1, 2, \ldots$), the remaining collection, if not empty, will be a chief collection of $H$ consisting of some or all of the permutations $s_1^p, s_2^p, \ldots, s_m^p$. Now $s_1^p, s_2^p, \ldots, s_m^p$ generate a proper subgroup of $H$, and therefore the chief collection remaining generates a proper subgroup $J$ of $H$, which is not allowable unless $J = e$ because, by hypothesis, $G$ is primitive. Hence after the removal of certain permutations because they occur in sets of $p$, only $e$, the identity, can remain. Thus $x^p = r$, where $r \neq e$, either has no solution in $K$ or $kp$ of the permutations $s_1, s_2, \ldots, s_m$ of $K$ satisfy it.

3. Wielandt's theorem. We are now prepared to prove

**Theorem 6.** If a simply transitive group $G$ not of prime degree contains a regular abelian subgroup $H$ of the same degree, and if one of the Sylow subgroups of $H$ is cyclic, $G$ is imprimitive.

Let $K$ be a primary subset (not $e$) of $H$, and let $p$ be the prime to which corresponds a cyclic Sylow subgroup of $H$. If $x_0$ is an element of $K$ such that $x_0^p = r$, and if $x_0y$ is another solution in $K$, $y^p = e$. But because $H$ has only one subgroup of order $p$, only $p$ permutations of $H$ satisfy $y^p = e$. If $P = \{s\}$ is this cyclic subgroup of order $p$, $x_0, x_0s, \ldots, x_0s^{p-1}$ are the only possible solutions of $x^p = r$ in $K$. Therefore, if $K$ contains no permutation of order $p$, and if $G$ is primitive, all the permutations of $K$ lie in the cosets $x_0P, x_1P, \ldots$, so that by Theorem 4 the group $G$ should be imprimitive. Since $H$ has at least three primary subsets, we can, when $p = 2$, choose $K$ so that $K \cap P$ is empty. In the following, then, $p$ is an odd prime. If $K$ is a subset of $P$, $K$ is singular because of the condition that $p$ is less than $n$; this again would make $G$ imprimitive.

If $G$ is primitive, $K = XP + F$, where the set $X$ consists of permutations $x_0, x_1, \ldots$ of $K$ whose $p$th powers are in $H - e$, and no two of which have the same $p$th power; and the set $F$ consists of the $f$ distinct permutations $s_1^q, s_2^q, \ldots, s_f^q$ of order $p$. Neither $X$ nor $F$ is empty. Now $P^2 = pP, PF = fP$, and in $F^2$ no permutation can occur more than $f$ times. Hence ($H$ being abelian) $K^2 = pX^2P + 2fXP + F^2$.

Let us say that we have chosen for our primary subset $K$ a primary subset of $H$ for which $f \leq (p - 1)/2$. Since $K^2$ is a chief collection and
2fXP contains a permutation of K repeated at least 2f times, K² contains 2fK = 2fXP + 2fF. Then K²−2fK is a chief collection and is equal to (pX²P + F²)−2fF. No element of F² occurs more than f times, so that every element of F is an element of X²P. If then from (pX²P + F²)−2fF we remove primary subsets p at a time until no permutation left occurs more than p−1 times, we get a chief collection F²+(p−2f)F, clearly singular, implying that G is imprimitive.

Thus Wielandt’s theorem is proved, and the proof is Wielandt’s, stripped to show its truly elementary character.

4. Linear substitutions. Let G be a transitive group on the n letters x₁, x₂, ⋯, xₙ. Let Gₖ be its subgroup that fixes the one letter xₖ. We agree as before that Gₖ is to be regarded as a group of degree n with r transitive constituents. Let one of the transitive constituents of G be on the m letters (or variables if you like) x₁₁, x₁₂, ⋯, x₁m, where x₁j is of course one of the letters x₁, x₂, ⋯, xₙ. Let ῦₖ be a permutation of G that replaces the m+1 letters x₁, x₁₁, x₁₂, ⋯, x₁m by xₖ, xₖ₁, xₖ₂, ⋯, xₖm, respectively. Then ῦₖ⁻¹Gₖ_ℌₖ = Gₖ, and in Gₖ is found a transitive constituent on xₖ₁, xₖ₂, ⋯, xₖm. Thus there is a one-to-one correspondence between the n subgroups G₁, G₂, ⋯, Gₙ and the n sets of letters x₁j, x₂j, ⋯, xₙj (j = 1, 2, ⋯, m). Then the linear substitution

\[
\begin{align*}
 x'_i &= x_{i1} + x_{i2} + \cdots + x_{im}, \\
 x'_2 &= x_{21} + x_{22} + \cdots + x_{2m}, \\
 & \quad \vdots \\
 x'_n &= x_{n1} + x_{n2} + \cdots + x_{nm}
\end{align*}
\]

(1)

is transformed into itself by every permutation of G. Neither the n subgroups Gₖ nor the n sets of m letters need be distinct. In this way r substitutions can be set up, and if each of them is multiplied by an arbitrary constant and they are then added, the result is a substitution commutative with every permutation of G. If m = 1, (1) is a permutation. If G is a regular group, r = n, and we get n distinct permutations, that is, the conjoin of G.

Let us suppose now that the substitution

\[ V: x'_i = \sum_{j=1}^{n} v_{ij} x_j, \quad i = 1, 2, \cdots, n, \]

is commutative with every permutation of G. Consider the first equation x'_i = \sum_{j=1}^{n} v_{ij} x_j of V. When V is transformed by all the permutations of G₁, x'_i is invariant and therefore the m coefficients of the
variables \( x_{11}, x_{12}, \ldots, x_{1m} \) must be equal, to \( v_1 \), say. Next, consider the transform of \( V \) by \( t_k \) (defined in the preceding paragraph): \( x'_1 \) goes into \( x'_k \), and the sum \( v_1(x_{11} + x_{12} + \cdots + x_{1m}) \) goes into \( v_1(x_{k1} + x_{k2} + \cdots + x_{km}) \), so that the variables \( x_{k1}, x_{k2}, \ldots, x_{km} \) in \( x'_k = \sum_{j=1}^{m} t_{k,j}^* x_j \) all have the same coefficient \( v_1 \), for \( k = 1, 2, \ldots, n \).

Hence:

**Theorem 7.** Every linear substitution commutative with each of the permutations of a transitive group can be obtained by first setting up the substitutions (1) and then multiplying them by appropriate constants and adding.

If the given transitive group \( G \) contains a regular subgroup \( H \) of the same degree \( n \), the permutations \( t_1 = e, t_2, t_3, \ldots, t_n \) can be the permutations of \( H \), and the substitution (1) will be the sum of \( m \) permutations of the conjoin of \( H \). There is a connection between the substitution (1) and a primary subset of \( H \).

**Theorem 8.** Let \( G \) be a transitive group of degree \( n \) and \( H \) a regular subgroup of order \( n \). Let the primary subset \( K \) of \( H \) be transformed into itself by every permutation of \( H \). Then the sum of the permutations of \( K \) is commutative with every permutation of \( G \).

This \( K \) is independent of our choice of subgroup fixing one letter. We return to the notations of §1. If \( H, H_1, \ldots, H_{w-1} \) is the complete set of conjugates to which \( H \) belongs under \( G \), it is also, because \( HG_x = G_x \), a complete set of conjugates under \( G_x \). Those permutations of \( G_x \) which transform \( H \) into \( H \), transform \( K \) into \( K \). Hence \( K \) has one and only one primary subset conjugate to it in each of the subgroups \( H, H_1, \ldots, H_{w-1} \). These \( w \) primary subsets \( K, K_1, \ldots, K_{w-1} \), whether distinct or not, constitute a conjugate set under \( G \). If in the permutations of \( K \) the letter \( x \) is followed by \( y_1, y_2, \ldots, y_m \), it is followed by the same \( m \) letters in some order in \( K_1 \), in \( K_2, \ldots, \) in \( K_{w-1} \). Therefore the sum of the permutations of \( K_i \) is equal to the sum of the permutations of \( K \) (\( i = 1, 2, \ldots, w-1 \)).

This result applies to every primary subset of \( H \) when \( H \) is abelian. The following theorem is more general.

**Theorem 9.** Let \( G \) be a transitive group of degree \( n \) and \( H \) a regular subgroup of order \( n \). Let \( G_a \) be the subgroup of \( G \) that fixes the one letter \( a \), and let \( K \) be a primary subset of \( H \) with respect to \( G_a \), composed of the \( m \) permutations

\[
t_1 = (c_1 a b_1 \cdots) \cdots, \ t_2 = (c_2 a b_2 \cdots) \cdots, \ldots, \ t_m = (c_m a b_m \cdots) \cdots.
\]
Let $K^*$ be the subset of the conjoin of $H$ composed of the $m$ permutations

$$t_1^* = (ac_1 \cdots) \cdots, t_2^* = (ac_2 \cdots) \cdots, \cdots,$$

$$t_m^* = (ac_m \cdots) \cdots.$$

Then the sum of the permutations of $K^*$ is a linear substitution commutative with every permutation of $G$.

There exists a permutation $(a)(b_1c_1)(b_2c_2) \cdots (b_mC_m) \cdots$ of order 2 which transforms the regular group $H$ into its conjoin $H^*$.

Any permutation $s$ that transforms $H$ into $H_i$, transforms $H^*$ into the conjoin $H_i^*$ of $H_i$; because if every permutation of $H^*$ is commutative with every permutation of $H$, then every permutation of $s^{-1}H^*s$ is commutative with every permutation of $s^{-1}Hs$. Let $N_a$ be the cross-cut of $G_a$ and $N$, the normalizer of $H$ in $G$. The permutations of $N_a$ transform each primary subset $K$ of $H$ with respect to $G_a$ into itself. The $w$ conjugate subgroups $H, H_1, \cdots, H_{w-1}$ contain $w$ primary subsets with respect to $G_a$: $K, K_1, \cdots, K_{w-1}$, forming, though perhaps not all distinct, a conjugate set under $G_a$, as do also the corresponding subsets $K^*, K_1^*, \cdots, K_{w-1}^*$ of $H^*, H_1^*, \cdots, H_{w-1}^*$ (respectively). The permutation $t = (ax \cdots) \cdots$ of $H$ transforms $G_a$ into $G_a$, and therefore replaces the letters $c_1, c_2, \cdots, c_m$ of a transitive constituent of $G_a$ by the letters $z_1, z_2, \cdots, z_m$ of a transitive constituent of $G_x$. But $t$ transforms each permutation of $K^*$ into itself. Hence in $K^*$, $x$ is followed by $z_1, z_2, \cdots, z_m$. Now

$$G_a = N_a + N_aq_1 + N_aq_2 + \cdots + N_aq_{w-1},$$

where each permutation of $N_aq_i$ transforms $K^*$ into $K_i^*$. The permutation $q_i$ is in $HG_x$, and every permutation of $H$ is commutative with every permutation of $K^*$, while every permutation of $G_x$ permutes among themselves the $m$ letters $z_1, z_2, \cdots, z_m$ that follow $x$ in the permutations of $K^*$. This being true for every letter $x$ of $G$, and for every $i$, the sum of the permutations of $K_i^*$ is equal to the sum of the permutations of $K^*$.

Stanford University